**Double-slit experiment with single electrons, “which-path” information and Dirac notation**

**The goal of this homework is to help you learn how:**

* scattering of a single electron by a photon in the vicinity of the slits can impact the electron that arrives at the screen
* the scattering of a single electron by a photon in the vicinity of the slits can modify the interference pattern
* the impact of the electron-photon scattering depends on the wavelength of the photon

The double slit experiment with a single electron source is pictured below. An electron source *s* emits electrons one at a time towards a plate with two slits and a fluorescent screen behind the plate serves as a detector and records the electrons that arrive at various positions on the screen (the interaction of an electron with the screen produces a flash of light).



What we want to know is whether an interference pattern will form on the screen after a large number of electrons have arrived at the screen. In order to determine this, we must answer the following question:

*What is the probability density that an electron emitted by the source arrives at some point x on the screen*?

To answer this question, we will make use of *three general rules of quantum mechanics*. *The first general rule* of quantum mechanics tells us that this probability density can be represented quantitatively by the absolute square of a complex number called the *probability density amplitude*. This amplitude can be represented using Dirac’s shorthand notation as:

$\left⟨ Electron leaves s\right⟩$.

This shorthand notation can be read as the “amplitude for an electron from the source *s* to arrive at position *x* on the screen”. In this expression written in Dirac notation, the expression to the right (electron leaves *s*) is the starting condition and the one on the left (electron arrives at *x*) is the final condition. In more condensed notation, this amplitude can be written as:

$\left⟨s\right⟩$.

where *s* stands for the source and *x* stands for a point on the screen.

An electron that arrives at position *x* on the screen could have gone through both of the two slits simultaneously. *The second general rule of quantum mechanics* tells us that when an electron can reach a given point *x* by two possible routes, the total amplitude for the process is the *sum of the amplitudes* for the two routes considered separately. Using Dirac notation, this can be written as:

$\left⟨s\right⟩\_{both slits open}=\left⟨s\right⟩\_{through slit 1}+\left⟨s\right⟩\_{through slit 2}$. (1)

1. Write down in your own words what the equation above says.

The amplitude for an electron from the source *s* to arrive at position *x* on the screen is equal to the sum of the amplitude for an electron from the source *s* to arrive at position *x* on the screen and go through slit 1 and the amplitude for an electron from the source *s* to arrive at position *x* on the screen and go through slit 2.

We now want to write out in more detail what we can say about the amplitude for the process in which an electron reaches the point *x* on the screen via slit 1. We do this by making use of the *third general rule*: When an electron goes by some particular route, the amplitude for that route can be written as the *product* of the *amplitude* to go part way with the *amplitude* to go the rest of the way. An electron that goes from the source to the screen via slit 1 has to go from the source to slit 1 and from slit 1 to the screen, therefore, $\left⟨s\right⟩\_{through slit 1}$ can be written as the product of the amplitude for the electron to go from the source to slit 1 and the amplitude for the electron to go from slit 1 to the screen:

$\left⟨s\right⟩\_{through slit 1}=\left⟨1\right⟩∙\left⟨s\right⟩$ (2)

Notice that equation (2) appears to be written in reverse order because it is to be read from right to left: an electron goes from the source to slit 1 and then from slit 1 to the point *x* on the screen.

2. In Dirac notation, write an expression similar to equation (2) for $\left⟨s\right⟩\_{through slit 2}$

$$\left⟨s\right⟩\_{through slit 2}=\left⟨2\right⟩∙\left⟨s\right⟩$$

3. Combine equations (1), (2) and the answer to question 2 to write an expression for $\left⟨s\right⟩\_{both slits open}$.

$$\left⟨s\right⟩\_{both slits open}=\left⟨1\right⟩\left⟨s\right⟩+\left⟨2\right⟩\left⟨s\right⟩$$

In the “which-path” information part of the double-slit tutorial you worked on, you learned that the wave function (probability density amplitude) of an electron at a point *x* on the screen (but before the interaction with the screen which makes it localized in position at the screen), in the case in which we do not have “which-path” information, is a linear superposition of the amplitudes from each slit, $Ψ(x)=Ψ\_{1}(x)+Ψ\_{2}(x)$. This is precisely what the equation you found in question 3 encapsulates. Therefore, we can identify that$ Ψ(x)=\left⟨s\right⟩\_{both slits open}$, $Ψ\_{1}(x)=\left⟨1\right⟩\left⟨s\right⟩$ and $Ψ\_{2}(x)=\left⟨2\right⟩\left⟨s\right⟩$.

Here, $Ψ\_{1}(x)$ and $Ψ\_{2}(x)$ are the amplitudes for an electron to go through each of the two slits (e.g., if one or the other slit is closed) and arrive at point *x* on the screen in the double slit experiment.

**Thought experiment designed to provide “which-path” information**

We will now examine a thought experiment (pictured below) designed to give us “which-path” information. A high-intensity light source *L* is placed between the slits and the screen and two photodetectors *D*1 and *D*2 are placed such that they detect photons scattered at slits 1 and 2 respectively. When an electron, on its way to the screen, scatters a photon either at slit 1 or at slit 2, the scattered photon will be recorded by one of the photodetectors (*D*1 or *D*2).



We make two major assumptions for this thought experiment:

1. An electron scatters off one photon only and this scattering occurs when the electron is at one of the slits and nowhere else.
2. The photodetectors only detect ALL the photons that are scattered by the electrons. They do not detect any photons coming from the light source that are not scattered by electrons.

This case is different from the case in which no light source was present because now scattering between an electron and a photon can occur and therefore the question we need to answer is the following:

*What is the amplitude for the process in which an electron starts at the electron source, scatters a photon from the light source (which is detected by one of the photodetectors D*1 *or D*2*) and the electron ends up at a position x on the screen?*

There are two possible final situations: electron arrives at position *x* / photon is detected by *D­*1 and electron arrives at position *x* / photon is detected by *D*2. Therefore, we must consider each case separately.

We first consider the case in which the photon is detected by *D*1:

* First, there is the amplitude $\left⟨s\right⟩$ that an electron goes from the source to slit 1.
* Then, we can suppose that there is some amplitude that while the electron is at slit 1, it scatters a photon which is recorded by the detector *D*1. Let us represent this amplitude, which is in general complex, by *a*.
* Then, there is the amplitude $\left⟨1\right⟩$ that the electron goes from slit 1 to the detector at point *x* on the screen.

4. Which of the following is the amplitude for the process in which an electron goes from the source *s* to point *x* on the screen via slit 1 *and* scatters a photon into detector *D*1 written in Dirac notation?

(A) $\left⟨1\right⟩\left⟨s\right⟩$

(B) $\left⟨1\right⟩a^{2}\left⟨s\right⟩$

(C) $\left|\left⟨1\right⟩\right|^{2}a\left|\left⟨s\right⟩\right|^{2}$

(D) $\left⟨1\right⟩a\left⟨s\right⟩$

(E) $\left⟨1\right⟩+a+\left⟨s\right⟩$

The correct answer to question 4 should include all three amplitudes that correspond to the three different processes that occur (electron goes from source to slit 1, at slit 1 the electron scatters a photon which is recorded by the detector *D*1 and the electron goes from slit 1 to the detector) and all the amplitudes should be to the first power and multiplied together.

5. How can the answer to question 4 be written in terms of $Ψ\_{1}(x)$ discussed earlier?

$$aΨ\_{1}(x)$$

The answer to question 4 is $\left⟨1\right⟩a\left⟨s\right⟩$, which is equal to $aΨ\_{1}(x)$.

Consider the following conversation between Andy and Caroline:

* Andy: We are done with the amplitude for a photon to scatter into detector *D*1. Therefore, now we should move on to finding the amplitude that an electron emitted from the source arrives at the screen and scatters a photon into detector *D*2.
* Caroline: But we have not considered the case in which an electron goes from the source to the screen via slit 2 *and* scatters a photon into detector *D*1.
* Andy: But it is impossible for a photon that gets scattered by an electron at slit 2 to be detected by detector *D*1. Detector *D*1 is close to slit 1, and would therefore only detect photons scattered at slit 1.
* Caroline: I disagree. Photons can behave as waves and if their wavelength is small enough, there can be significant diffraction effects. Therefore, it is possible that a photon scattered at slit 2 is detected by detector *D*1, but this probability should be much smaller than the probability that a photon scattered at slit 2 is detected by detector *D*2. Nevertheless, in order to keep the discussion general, we must take into account that there is always some such probability amplitude.

With whom, if either, do you agree? Explain.

Caroline is correct.

Caroline is correct, so in order to keep the discussion general, we represent the amplitude that while an electron is at slit 2, it scatters a photon into photodetector *D*1 by *b* (*b* is, in general, complex). Because *a* and *b* are amplitudes that an electron scatters off a photon either at slit 1 or at slit 2, $\left|a\right|^{2}+\left|b\right|^{2}=1$.

6. Write the amplitude that an electron goes via slit 2 *and* scatters a photon into photodetector *D*1, first in Dirac notation (similar to the answer of question 4) and then in terms of $Ψ\_{2}(x)$ (similar to the answer of question 5).

$$\left⟨2\right⟩b\left⟨s\right⟩= bΨ\_{2}(x)$$

You should have found that the two probability amplitudes are $aΨ\_{1}(x)$ and $bΨ\_{2}(x)$, which correspond to the only two ways in which a photon can be scattered into photodetector 1.

7. Fill in the blank (in terms of $Ψ\_{1}(x)$ and $Ψ\_{2}(x)$):

$$\left⟨\begin{matrix}electron from s\\photon from L\end{matrix}\right⟩=aΨ\_{1}\left(x\right)+bΨ\_{2}(x)$$

8. Write in your own words what the amplitude above represents.

This is the amplitude for the process in which an electron leaves the source *s* and arrives at point *x* on the screen, and a photon is detected by photodetector *D*1.

You will now use the symmetry of the double slit setup to find the amplitude for an electron to arrive at point *x* on the screen and a photon to be registered in detector *D*2.

9. Complete the following sentence correctly: “The amplitude that an electron at slit 1 scatters a photon into detector 1 is

(A) larger

(B) smaller

(C) equal to

the amplitude that an electron at slit 1 scatters a photon into detector 2.”

10. What is the amplitude for when an electron is at slit 1, it scatters a photon into detector *D*2?

(a) $a$

(b) $b$

(c) $a+b$

(d) $\left|a\right|^{2}+\left|b\right|^{2}$

Explain your reasoning.

11. Write down the amplitude for an electron to go from the source to the screen at point *x* through slit 1 and scatter a photon into detector *D*2 both in Dirac notation (i.e., something similar to $\left⟨2\right⟩b\left⟨s\right⟩$ ) and using $Ψ\_{1}(x)$ and $Ψ\_{2}(x)$.

$$\left⟨1\right⟩b\left⟨s\right⟩=bΨ\_{1}(x)$$

12. What is the amplitude for when an electron is at slit 2, it scatters a photon into detector *D*2?

(a) $a$

(b) $b$

(c) $a+b$

(d) $\left|a\right|^{2}+\left|b\right|^{2}$

Explain your reasoning.

Due to the symmetry of the set-up, this amplitude must equal the amplitude for when an electron is at slit 1, it scatters a photon into detector *D*1.

13. Write down the amplitude for an electron to go from the source *s* to the screen at point *x* through slit 2 and scatter a photon in detector *D*2 both in Dirac notation and in terms of $Ψ\_{1}(x)$ and $Ψ\_{2}(x)$.

$$\left⟨2\right⟩a\left⟨s\right⟩=aΨ\_{2}(x)$$

14. In your own words, write down what the following expression represents?

 $\left⟨\begin{matrix}electron from s\\photon from L\end{matrix}\right⟩$

This is the amplitude for the process in which an electron leaves the source *s* and arrives at point *x* on the screen, and a photon is detected by photodetector *D*2.

15. Fill in the blank (using $a$, $b$, $Ψ\_{1}(x)$ and $Ψ\_{2}(x)$):

$\left⟨\begin{matrix}electron from s\\photon from L\end{matrix}\right⟩=$ $bΨ\_{1}\left(x\right)+aΨ\_{2}(x)$

Consider the following conversation between Andy and Caroline:

● Andy: To find the total probability density for an electron to arrive at point *x* on the screen and a photon to be registered at either detector *D*1or *D*2, we add the two probability amplitudes we found and take the absolute square.

● Caroline: No, we cannot add those two probability amplitudes because they do not correspond to the same final state. In one case, the electron is at position *x* on the screen and the photon is detected by photodetector 1 and in the other case the electron is at position *x* and the photon is detected by photodetector 2. Because the two final states are different, we have to first take the absolute square of the amplitudes that correspond to each state and then add them.

With whom do you agree?

(a) Andy

(b) Caroline

(c) Neither

Explain.

16. Write down the probability density for an electron to arrive at point *x* on the screen and a photon to be registered at either detector *D*1or *D*2 in terms of $a$, $b$, $Ψ\_{1}(x)$ and $Ψ\_{2}(x)$. Do NOT try to simplify the initial answer you obtain!

(Hint: you must square the two probability density amplitudes individually and then add).

$$\left|aΨ\_{1}\left(x\right)+bΨ\_{2}(x)\right|^{2}+\left|aΨ\_{2}\left(x\right)+bΨ\_{1}(x)\right|^{2}$$

The answer to question 16 is $\left|aΨ\_{1}\left(x\right)+bΨ\_{2}(x)\right|^{2}+\left|aΨ\_{2}\left(x\right)+bΨ\_{1}(x)\right|^{2}$.

The following questions are intended to help you make the connection between what you learned working on this homework and what you learned in the double slit tutorial about “which-path” information.

In all of the following questions, assume that the photon source has high intensity such that each electron scatters off a photon.

17. In the following eight questions (17. I. – 17. VIII.), you will examine the case $b=0$.

17. I. Consider an electron emitted by the source that is detected at the screen, and in the process of the electron going from the source to the screen, it scatters a photon into *D*1. Does the scattering process provide “which-path” information for the electron? Explain your reasoning.

If *b* = 0, then *a* = 1. In other words, all the photons scattered at slit 1 are detected by photodetector *D*1 and all the photons scattered at slit 2 are detected by photodetector *D*2. Therefore, if in the process of the electron going from the source to the screen, it scatters a photon into *D*1, this scattering between the electron and the photon occurred at slit 1. Consequently, after the electron scatters off the photon, we have “which-path” information for the electron.

17. II. Consider an electron emitted by the source that is detected at the screen, and in the process of the electron going from the source to the screen, a photon is scattered into *D*2. Does the scattering process provide “which-path” information for the electron? Explain your reasoning.

If *b* = 0, then *a* = 1. In other words, all the photons scattered at slit 1 are detected by photodetector *D*1 and all the photons scattered at slit 2 are detected by photodetector *D*2. Therefore, if in the process of the electron going from the source to the screen, it scatters a photon into *D*2, this scattering between the electron and the photon occurred at slit 2. Consequently, after the electron scatters off the photon, we have “which-path” information for the electron.

17. III. Do you expect to observe an interference pattern on the screen after a large number of electrons arrive at the screen in the case *b* = 0? Explain your reasoning.

No, we do not expect to observe an interference pattern in the case *b* = 0, because in this case we have “which-path” information for all electrons after they scatter. The interference pattern is destroyed if “which-path” information can be obtained.

17. IV. Look back at your answer in question 16. What is the probability density when *b* = 0? Simplify your answer.

$$\left|Ψ\_{1}\left(x\right)\right|^{2}+\left|Ψ\_{2}\left(x\right)\right|^{2}$$

17. V. Are there any cross terms in the probability density you wrote in the preceding question for the case *b* = 0? Do you expect to observe an interference on the screen? If your answer to this question is inconsistent with your answer to 17.III., can you reconcile the difference?

There are no cross terms in the expression above and therefore we do not expect to observe an interference pattern. This is consistent with the answer to 17.III.

17. VI. Based on your responses to the previous questions (17. I – 17. V), how does the wavelength of the photon compare to the distance between the slits for the case *b* = 0? Explain your reasoning.

In the case *b* = 0, we do not observe an interference pattern. Therefore, scattering of an electron off a photon at the slits provides “which-path” information for the electron. This is only true if the wavelength of the photons is significantly smaller than the distance between the slits.

17. VII. Explain in your own words the connection between *b* = 0, whether or not we obtain “which-path” information, and how the wavelength of the photon compares to the distance between the slits (i.e., summarize your previous six answers).

In the case *b* = 0, *a* = 1 and therefore all the photons collected by *D*1 correspond to scattering events which occurred at slit 1 and all the photons collected by *D*2 correspond to scattering events which occurred at slit 2. We have “which-path” information for all the electrons because scattering with photons localizes them to be at one slit or the other. The length scale of this localization, which is comparable to the wavelength of the photon must be significantly smaller than the distance between the slits. Therefore, the wavelength of the photon is significantly smaller than the distance between the slits.

18. In the following eight questions (18. I. – 18. VIII.), you will examine the case $a=b$.

18. I. Consider an electron emitted by the source that is detected at the screen, and in the process of the electron going from the source to the screen, it scatters a photon into *D*1. Does the scattering process provide “which-path” information for the electron? Explain your reasoning.

If *a* = *b*, there is an equal probability that a photon detected by photodetector *D*1 was scattered at slit 1 or at slit 2 (the same is true for the case in which a photon is detected by *D*2). In other words, from the point of view of the photon, it is very difficult to distinguish between the two slits. Therefore, scattering of an electron and a photon does not provide “which-path” information for the electron.

18. II. Consider an electron emitted by the source that is detected at the screen, and in the process of the electron going from the source to the screen, it scatters a photon into *D*2. Does the scattering process provide “which-path” information for the electron? Explain your reasoning.

If *a* = *b*, there is an equal probability that a photon detected by photodetector *D*1 was scattered at slit 1 or at slit 2 (the same is true for the case in which a photon is detected by *D*2). In other words, from the point of view of the photon, it is very difficult to distinguish between the two slits. Therefore, scattering of an electron and a photon does not provide “which-path” information for the electron.

18. III. Do you expect to observe an interference pattern on the screen after a large number of electrons arrive at the screen in the case *a* = *b*? Explain your reasoning.

Yes, we do expect to observe interference in this case because we do not have “which-path” information and the electrons go through both slits at the same time.

18. IV. Look back at your answer to question 16. What is the probability density when *a* = *b*? Simplify your answer.

($\left|a\right|^{2}+\left|b\right|^{2}=1)$

$$\left|Ψ\_{1}\left(x\right)\right|^{2}+\left|Ψ\_{2}\left(x\right)\right|^{2}+Ψ\_{1}\left(x\right)Ψ\_{2}^{\*}\left(x\right)\left(ab^{\*}+ba^{\*}\right)+Ψ\_{2}\left(x\right)Ψ\_{1}^{\*}\left(x\right)\left(ab^{\*}+ba^{\*}\right)$$

18. V. Are there any cross terms in the expression you wrote in the preceding question for the case *a* = *b*? Do you expect to observe interference on the screen?

The last two terms in the expression above have $Ψ\_{1}\left(x\right)Ψ\_{2}^{\*}\left(x\right)$ and $Ψ\_{2}\left(x\right)Ψ\_{1}^{\*}\left(x\right)$ which are cross terms that will be responsible for an interference pattern on the screen. Therefore, yes, we do expect to observe interference on the screen.

18. VI. Based on your responses to the previous questions (18. I – 18. V), how does the wavelength of the photon compare to the distance between the slits in the case *a* = *b*? Explain your reasoning.

In the case *a* = *b*, we do not obtain “which-path” and observe an interference pattern on the screen. This is only true when the wavelength of the photons emitted by the light source is significantly larger than the distance between the slits.

18. VII. Explain in your own words the connection between *a* = *b*, whether or not we obtain “which-path” information, and how the wavelength of the photon compares to the distance between the slits (i.e., summarize your previous seven answers).

If *a* = *b*, there is an equal probability that a photon detected by photodetector *D*1 was scattered at slit 1 or at slit 2 (the same is true for the case in which a photon is detected by *D*2). This means that from the point of view of the photons, it is difficult to distinguish between the two slits (they overlap) and a photodetector cannot be used to differentiate between them. Therefore, scattering between an electron and a photon at the slits does not provide “which-path” information. In order for this to be true, the wavelength of the photon is significantly larger than the distance between slits.

Alternatively, one can think of the interaction between a photon and an electron which localizes the electron within a region of length scale comparable to the distance between the slits. The length scale of the localization is so large compared to the distance between the slits that the probability that scattering occurred at one slit is identical to the probability that it occurred at the other. Therefore, the length scale of the localization (wavelength of the photon) must be significantly larger than the distance between the slits.

You will now examine the case $a=0$ to determine whether it makes physical sense or not. Just like in questions 17 and 18, assume that the lamp is turned on.

19. I. If $a=0$, what is the probability that a photon scattered at slit 1 is detected by *D*1?

0%

19. II. If $a=0$, what is the probability that a photon scattered at slit 1 is detected by *D*2?

100%

19. III. Does the case $a=0$ make physical sense? Why or why not?

The previous two answers indicate that the probability that a photon scattered at slit 1 is detected by photodetector *D*1is 0%, while the probability that a photon scattered at slit 1 is detected by photodetector *D*2 is 100%. This makes no physical sense because photodetector *D*1 is closer to slit 1 and “aimed” at it, while photodetector *D*2 is farther from slit 1 and “aimed” at slit 2. Therefore, the probability that a photon scattered at slit 1 is detected by *D*1 must be larger than (or equal to) the probability that it will be detected by *D*2 (i.e., $\left|a\right|^{2}\geq \left|b\right|^{2}$).