**Bloch sphere**

**Notes:**

- In the Bloch sphere diagrams of this section, the $+x$, $+y$, and $+z$ axes point toward the viewer, while the $-x$, $-y$, and $-z$ axes point away from the viewer.
- Solid lines indicate states on the front hemisphere (which point toward the viewer), while dotted lines indicate states on the back hemisphere.
- When a measurement basis is indicated, the larger ket label indicates the closer state (which points toward the viewer), while the smaller ket label indicates the state further away.

- In these questions, states are said to be “equivalent” if they yield identical measurement outcomes with identical probabilities for all observables (i.e., in all measurement bases).
- A state $|q\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle$ can be found on the Bloch sphere at the position of the Cartesian vector $\mathbf{r} = \sin \theta \cos \phi \mathbf{i} + \sin \theta \sin \phi \mathbf{j} + \cos \theta \mathbf{k}$, where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.
  - The polar angle $\theta$ begins from the positive $z$-axis and sweeps toward the equator, and the azimuthal angle $\phi$ begins from the positive $x$-axis and sweeps counterclockwise about the $z$-axis.
Warm-up

1. Consider the states \( |q\rangle = \frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle \) and \( |p\rangle = \frac{i\sqrt{3}}{2} |0\rangle + \frac{i}{2} |1\rangle \). Are \( |q\rangle \) and \( |p\rangle \) equivalent states, i.e., do they yield identical measurement outcomes with identical probabilities in all bases?
   Yes. They differ by an overall phase, which is not measurable.

2. Consider the states \( |b\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{i}{\sqrt{2}} |1\rangle \) and \( |d\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{i}{\sqrt{2}} |1\rangle \). Are \( |b\rangle \) and \( |d\rangle \) equivalent states?
   No. They differ by a relative phase rather than an overall phase. (The states give the same measurement outcomes in the \( \{ |0\rangle, |1\rangle \} \) basis, but not in other bases.

3. True or false: Any normalized state written as \( a |0\rangle + b |1\rangle \) can be found on the Bloch sphere with a unique \( \theta \) and \( \phi \).
   True

Measurements

4. Consider the state \( |q\rangle = \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} e^{i\frac{2\pi}{3}} |1\rangle \).
   (A) On the Bloch sphere below, indicate the state \( |q\rangle \). Label \( \theta \) and \( \phi \).

   \[ \theta = \frac{\pi}{8} \]
   \[ \phi = \frac{2\pi}{3} \]

   (B) The state \( |q\rangle = \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} e^{i\frac{2\pi}{3}} |1\rangle \) is measured in the \( \{ |0\rangle, |1\rangle \} \) basis. What are the outcomes? What is the probability of yielding each outcome? (There is no need to find numerical values.)
   \[ \text{Probability of } |0\rangle \text{ is } \left| \cos \frac{\pi}{8} \right|^2 = \left( \cos \frac{\pi}{8} \right)^2 \]
   \[ \text{Probability of } |1\rangle \text{ is } \left| \sin \frac{\pi}{8} e^{i\frac{2\pi}{3}} \right|^2 = \left( \sin \frac{\pi}{8} \right)^2 \left( e^{i\frac{2\pi}{3}} \right)^2 \]

   (C) The state \( |q\rangle \) is measured in the \( \{ \langle + \rangle, \langle - \rangle \} \) basis. Which outcome is more probable? \( \langle - \rangle \). (The state \( |q\rangle \) “leans” closer to the \( \langle - \rangle \) state than the \( \langle + \rangle \) state.)

   (D) The state \( |q\rangle \) is measured in the \( \{ \langle +i \rangle, \langle -i \rangle \} \) basis. Which outcome is more probable? \( \langle +i \rangle \). (The state \( |q\rangle \) “leans” closer to the \( \langle +i \rangle \) state than the \( \langle -i \rangle \) state.)
5. Consider a qubit in the state $|q⟩ = a|0⟩ + b|1⟩$ shown below.

$|q⟩$ is illustrated in the following three diagrams, each one highlighting a different measurement basis. The (nonstandard) notations $θ_1$, $θ_2$, and $θ_3$ will be used to refer to the angles made with respect to one of the states in this measurement bases.

(A) In diagram (I), $|q⟩$ makes an angle $θ_1$ with the $|0⟩$ state.
(B) In diagram (II), $|q⟩$ makes an angle $θ_2$ with the $|+⟩$ state.
(C) In diagram (III), $|q⟩$ makes an angle $θ_3$ with the $|+i⟩$ state.

Express your answers to (A) and (B) in terms of $θ_1$, $θ_2$, and $θ_3$.

(A) When $|q⟩$ is measured in the $\{|+⟩, |−⟩\}$ basis, what is the probability that $|q⟩$ will collapse into the state $|+⟩$?

$$\left(\cos \frac{θ_1}{2}\right)^2 = \left\{\sin \left(\frac{π - θ_1}{2}\right)\right\}^2$$

(B) When $|q⟩$ is measured in the $\{|+i⟩, |−i⟩\}$ basis, what is the probability that $|q⟩$ will collapse into the state $|−i⟩$? (Note: NOT $|+i⟩$!)

$$\left(\sin \frac{θ_3}{2}\right)^2 = \left\{\cos \left(\frac{π - θ_3}{2}\right)\right\}^2$$
6. Below is shown a cross-section of the Bloch sphere, with the axes indicated.

(A) On the cross-section of the Bloch sphere, indicate a state of your choosing that, when measured in the \{\vert 0 \rangle, \vert 1 \rangle \} basis, has a higher chance of yielding \vert 0 \rangle than \vert 1 \rangle.

(B) On the cross-section of the Bloch sphere, indicate a state of your choosing that, when measured in the \{\vert 0 \rangle, \vert 1 \rangle \} basis, has a higher chance of yielding \vert 1 \rangle than \vert 0 \rangle.

(C) On the cross-section of the Bloch sphere, indicate a state of your choosing that, when measured in the \{\vert + \rangle, \vert - \rangle \} basis, has a higher chance of yielding \vert + \rangle than \vert - \rangle.

[Sample answers. The indicated state just needs to be closer to the state that it has a higher chance of yielding.]

**Geometric intuition**

7. On the Bloch sphere below, the \{\vert q \rangle, \vert -q \rangle \} basis is illustrated. Indicate (or otherwise specify) all the states in which a measurement in this basis yields a result with 100% certainty.

\( \vert q \rangle \) and \( \vert -q \rangle \) are the only such states.
8. Consider the state \( |p\rangle \) on the Bloch sphere below.

You want to make a measurement in the state \( |p\rangle \) such that you get a certain outcome with 100% probability. How many measurement bases can you choose to accomplish this?

a. 0  

b. 1  

c. 2  

d. Infinitely many

9. On the Bloch sphere below, indicate the set of all states that, when measured in the illustrated \( \{|q\rangle, |-q\rangle\} \) basis, will yield \( |q\rangle \) with (approximately) 90% probability and \( |-q\rangle \) with 10% probability.

(a circle on the surface of the Bloch sphere centered on \( |q\rangle \))

Strictly speaking,

\[
\left( \cos \frac{\theta}{2} \right)^2 = 0.9
\]