Warm-up for Stern-Gerlach & Larmor Precession Tutorials

1. Which one of the following statements is true about the relationship between the magnitude of the orbital angular momentum $\vec{L}$ of a particle and the corresponding magnetic dipole moment $\vec{\mu}$?

A. $\vec{\mu} = \gamma \vec{L}$, where γ is the gyromagnetic ratio.

B. $|\vec{\mu}| = \dfrac{\gamma}{|\vec{L}|}$

C. $\vec{\mu}$ is independent of $\vec{L}$

2. Choose all of the following physical quantities on which the gyromagnetic ratio γ of an electron depends.
   (1) mass m (2) charge e (3) energy E

A. 1 only
B. 2 only
C. 3 only
D. 1 and 2 only
E. 2 and 3 only

3. A student says that, similar to a magnetic dipole moment associated with the orbital angular momentum $\vec{L}$, there will be a magnetic dipole moment associated with spin $\vec{S}$. Explain why you agree or disagree. If you agree, write the spin magnetic dipole moment $\vec{\mu}_s$ in terms of γ and $\vec{S}$.

4. $\vec{\mu}$ is the magnetic dipole moment of a classical particle. If the particle is in a uniform external magnetic field $\vec{B}$, the magnetic field would exert a torque $\vec{\mu} \times \vec{B}$ on the particle. Choose all of the following statements that are correct.
   (1) The torque tends to line up the magnetic dipole moment **perpendicular** to the field.
   (2) The torque would cause the Larmor precession of the magnetic dipole moment about the field.

A. 1 only
B. 2 only
C. Both 1 and 2
D. Neither 1 nor 2

5. For a particle with a magnetic dipole moment $\vec{\mu}$ at rest in an external magnetic field, the energy associated with the magnetic dipole moment is $U = -\vec{\mu} \cdot \vec{B}$, which is also the Hamiltonian of the system. If the non-zero magnetic field $\vec{B}$ is <u>uniform</u>, choose all of the following statements that are correct.
   (1) The torque on the particle is non-zero.
   (2) The force on the particle is non-zero.

A. only 1
B. only 2
C. both 1 and 2
D. neither of them

6. Particles with a "net" charge q, e.g., electrons, are sent through the Stern-Gerlach apparatus (SGA). Choose all of the forces that the particles could possibly feel in the magnetic field.
   (1) The Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$
   (2) The force $\vec{F} = -\nabla U = \nabla(\vec{\mu} \cdot \vec{B})$ due to the magnetic dipole moment, which is related to the angular momentum.

A. only 1
B. only 2
C. 1 and 2
D. Either 1 or 2 but not both.

7. Choose all of the following magnetic fields that can be used in the Stern Gerlach experiment to spatially separate the different components of angular momentum of an atom. (Ignore the fact that some $\vec{B}$ below may violate the electromagnetic law $\vec{\nabla} \cdot \vec{B} = 0$.)

(1) $\vec{B} = C_0 \hat{k}$

(2) $\vec{B} = C_0 z \hat{k}$

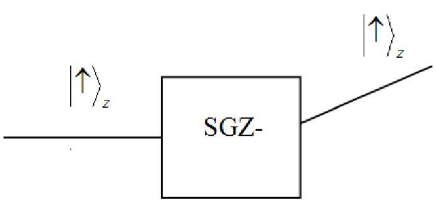
(3) $\vec{B} = C_1 \hat{i} + C_2 \hat{j}$

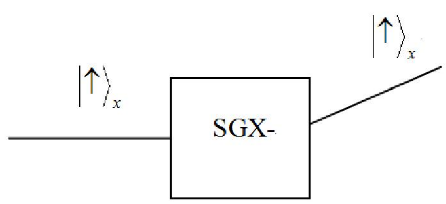
(4) $\vec{B} = C_1 \hat{i} + C_2 y^2 \hat{j}$

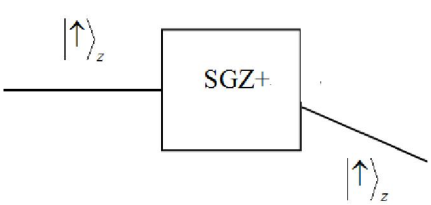
A. 1 and 2
B. 2 and 3
C. 2 and 4
D. 2, 3 and 4

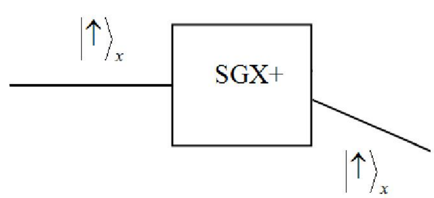
The magnetic field $\vec{B}$ should satisfy the electromagnetic law $\vec{\nabla} \cdot \vec{B} = 0$ so the non-uniform magnetic field used in the Stern-Gerlach experiment should have inhomogeneous components in at least two directions, e.g., $\vec{B} = -\alpha x\hat{i} + (B_0 + \alpha z)\hat{k}$. However, the strong uniform magnetic field $B_0\hat{k}$ causes the Larmor precession about the *z*-axis so the *x*-component of the spin angular momentum $\hat{S}_x$ averages to zero. Therefore, for the magnetic field $\vec{B} = -\alpha x\hat{i} + (B_0 + \alpha z)\hat{k}$, we can neglect its *x*-component and only consider the magnetic field gradient in the *z*-direction.

*In the following questions, **SGZ+ (SGZ-)** denotes a Stern-Gerlach apparatus (**SGA**) with magnetic field gradient in **+z (-z)** direction. Similarly we can define SGX+, SGX-, SGY+ and SGY-.* If an atom with state $\left|\uparrow\right\rangle_z$ (or $\left|\downarrow\right\rangle_z$) passes through SGZ-, it will be deflected in the +z (or -z) direction. If an atom with state $\left|\uparrow\right\rangle_z$ (or $\left|\downarrow\right\rangle_z$) passes through SGZ+, it will be deflected in the -z (or +z) direction. Similarly, if an atom with state $\left|\uparrow\right\rangle_x$ passes through SGX- (or SGX+), it will be deflected in the +x (or -x) direction. The figures below show examples of deflections through the SGX and SGZ in the plane of the paper. However, the deflection through a SGX will be in a plane perpendicular to the deflection through an SGZ. This actual three-dimensional nature should be kept in mind in the tutorial.









8. Classically, we use "force" and "torque" to analyze the movement of the particle. However, in quantum mechanics, we use the Hamiltonian to determine the behavior of a quantum system. In an external magnetic field $\vec{B}$, the energy associated with the spin magnetic dipole moment (or spin angular momentum) is determined by a corresponding Hamiltonian operator $\hat{H} = -\vec{\mu} \cdot \vec{B}$ for the spin degree of freedom. In the SGZ+, the

particle enters a non-uniform magnetic field $\vec{B} = C_0 z\hat{k}$ ($\hat{k}$ is the unit vector in $+z$ direction and C_0 is a constant) at time $t = 0$ and exits at $t = T$. Which one of the following is the correct expression for $\hat{H}$? γ is a constant for all the options.

A. $\hat{H} = -C_0 \gamma \hat{S}_z \hat{k}$ for $0 \le t \le T$ and 0 otherwise.

B. $\hat{H} = -C_0 \gamma \hat{S}_z$ for $0 \le t \le T$ and 0 otherwise.

C. $\hat{H} = -C_0 \gamma z \hat{S}_z \hat{k}$ for $0 \le t \le T$ and 0 otherwise.

D. $\hat{H} = -C_0 \gamma z \hat{S}_z$ for $0 \le t \le T$ and 0 otherwise.

9. Show your work for how you arrived at the answer to the previous question (*question 8*).

10. A silver atom in its ground state has an orbital angular momentum quantum number $\ell = 0$ and a spin angular momentum quantum number $s = 1/2$. Which one of the following matrices correctly represents the Hamiltonian $\hat{H}$ for the spin degrees of freedom in a non-uniform magnetic field $\vec{B} = C_0 z\hat{k}$? For reference, if we choose the eigenstates of $\hat{S}_z$ as the basis vectors, the components of the spin angular momentum are given by: $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

A. $-\gamma C_0 z \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

B. $-\gamma C_0 z \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

C. $-\gamma C_0 z \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

D. $-\gamma C_0 z \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

11. If the Hamiltonian operator $\hat{H}$ commutes with a particular component of spin, we can find a complete set of simultaneous eigenstates of both of them. What are the normalized eigenstates of $\hat{H}$ in *question 10*? Are they the same as the normalized eigenstates of $\hat{S}_z$? Explain.

12. Choose the correct representation for the energy eigenvalues E_+ and E_- of the Hamiltonian in *question 10*?

A. $\mp C_0 \gamma \hbar / 2$

B. $\mp C_0 \gamma / 2$

C. $\mp C_0 \gamma z \hbar / 2$

D. $\mp C_0 \gamma z / 2$

13. Suppose the eigenvalues of the Hamiltonian operator $\hat{H}$ are E_+ and E_-. The initial state of the silver atom at time t=0 right before entering the SGZ- is $a|\uparrow\rangle_z + b|\downarrow\rangle_z$. Write down the state of the atom at time $t = T$ right after exiting the SGZ-?

14. If we choose $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$, the orthonormal eigenstates of $\hat{S}_z$, as the basis vectors for the two dimensional spin space, which of the following statements correctly represents a general spin state $|\chi\rangle$?

A. $|\chi\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, where $|a|^2 + |b|^2 = 1$

B. $|\chi\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, where a and b can be any complex numbers.

C. $|\chi\rangle = a|\uparrow\rangle_z \times b|\downarrow\rangle_z$, where a and b can be any complex numbers.

15. A silver atom in the spin state $\chi(t = 0) = (a|\uparrow\rangle_z + b|\downarrow\rangle_z)$ at time t=0 passes through an SGZ- with a non-uniform magnetic field $\vec{B} = C_0 z\hat{k}$ from time t=0 to t=T. Which one of the following is the spin state at time t=T when the atom just exits the magnetic field? (Hint: The time development of each stationary state is via an appropriate term of the type

$e^{\pm iE_{\pm}t/\hbar}$)

A. $\chi(T) = a\phi_+|\uparrow\rangle_z + b\phi_-|\downarrow\rangle_z$, where $\phi_\pm = e^{\pm iC_0\gamma \cdot zT/2}$

B. $\chi(T) = \phi_+(a|\uparrow\rangle_z + b|\downarrow\rangle_z)$

C. $\chi(T) = (a|\uparrow\rangle_z + b|\downarrow\rangle_z)$

D. None of the above.

16. Silver atoms in an initial normalized spin state $\chi(0) = \frac{3}{5}|\uparrow\rangle_z + \frac{4}{5}|\downarrow\rangle_z$ pass one at a time through a SGZ-. According to your answer to the previous question (*question 15*), write down the state of an atom after it passes through the SGZ.

17. In the previous problem (*question 16*), if you place a single detector in the path of the "spin-up" component, which one of the following correctly gives the probability that the detector would click?

A. Every time an atom is sent through the SGA, the detector will click.

B. Every time an atom is sent through the SGA, the detector has a 3/5 probability of clicking.

C. Every time an atom is sent through the SGA, the detector has a 9/25 probability of clicking.

D. The detector will never click.

In question 17, does the probability depend on time? Explain.

The force $\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}) = C_0\gamma \cdot S_z$ is determined by the Hamiltonian and the actual form of the external magnetic field as we discussed earlier. However, note that the "force" on $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ components of the state will be in opposite direction because they couple to momentum in opposite direction. Therefore, these components will get spatially separated over time. When the initial spin state is $a|\uparrow\rangle_z + b|\downarrow\rangle_z$, after time T through the SGZ the spin state would become $\chi(t) = a \cdot e^{iE_+t/\hbar}|\uparrow\rangle_z + b \cdot e^{iE_-t/\hbar}|\downarrow\rangle_z$. On the other hand, if the initial spin state is $|\uparrow\rangle_z$, it only couples with upward momentum as $e^{iE_+t/\hbar}|\uparrow\rangle_z$. These correlation should be kept in mind when determining where to place the detectors.

18. In the wavefunction $\chi(T) = a\phi_+ \left|\uparrow\right\rangle_z + b\phi_- \left|\downarrow\right\rangle_z$, the spatial and spin parts of the wave functions are coupled. The two phase factors $\phi_\pm$ can be expressed as $\phi_\pm = e^{\pm ik_z z}$ where $k_z = \pm C_0 \gamma T/2$. Thus, $e^{\pm ik_z z}$ correspond to the two spin components having momenta in the $\pm z$ directions. Recall that the z-component of momentum is $p_z = \hbar k_z$ (from deBroglie relation) and $e^{\pm ik_z z}$ represent momentum eigenstates (plane waves propagating in the $\pm z$ direction). Choose all of the following statements that are true right after an atom has passed through SGZ:

(1) Since the eigenstates of $\hat{S}_z$ in the wave function carry z-component of momenta in opposite directions, they will become spatially separated.

(2) The state $\chi(T)$ will remain in a linear superposition of $\left|\uparrow\right\rangle_z$ and $\left|\downarrow\right\rangle_z$ until a measurement is performed. Simply passing through the SGA would not collapse the spin state to $\left|\uparrow\right\rangle_z$ and $\left|\downarrow\right\rangle_z$

(3) As soon as the atom leaves the magnetic field, its wave function should go back to the original state before passing through the SGA.

A. 1 only B. 2 only C. 1 and 2 only D. 2 and 3 only E. All of the above

→[explanation] ***The two components of spin in the wave function become spatially further apart after the SGA, but the wavefunction remains in a superposition of spin states unless a measurement is performed (e.g., by placing a detector in front of a component of beam which collapses the wavefunction).***

19. A large number of atoms pass through the SGZ- and are in the state $\Psi(T) = a\phi_+ \left|\uparrow\right\rangle_z + b\phi_- \left|\downarrow\right\rangle_z$. Two detectors placed in suitable locations immediately after the SGZ- register these atoms. Choose all of the following statements that are correct:

(1) Since the wavefunction couples the momentum to the spin degree of freedom, the clicking of a particular detector informs us about the spin state of each atom after measurement.

(2) Depending on which detector clicks, the state of the system collapses to either $\left|\uparrow\right\rangle_z$ or $\left|\downarrow\right\rangle_z$.

(3) After clicks are registered from a large number of atoms, the values of $|a|^2$ and $|b|^2$ in the wave function before the collapse can be inferred.

A. 1 only B. 3 only C. 1 and 2 only D. 2 and 3 only E. All of the above

20. In the Stern-Gerlach experiment, suppose a screen is placed after the magnetic field. When a particle hits the screen, it produces a spot on the screen. Choose all of the following statements that are correct.
    (1) A beam of neutral silver atoms passing through a uniform magnetic field will leave two clusters of spots on the screen.
    (2) A beam of neutral silver atoms passing through a non-uniform magnetic field may leave two clusters of spots on the screen under suitable condition.
    (3) A beam of silver ions (in which the only valence electron that silver has is knocked out) passing through the non-uniform magnetic field will leave two well-separated clusters of spots on the screen.

A. only 1
B. only 2
C. 1 and 3
D. 2 and 3
E. All of the above

→[explanation] ***The spin quantum number of a silver atom is 1/2, which is due to the spin of the outermost electron. When we knocked out the valence electron in a silver atom, the spin of the silver ion is zero. So the silver ion beam would not separate into two parts.***

For a neutral particle with magnetic dipole moment $\vec{\mu}$ ***interacting with a uniform magnetic field, the force on the particle is*** $\nabla(\vec{\mu}\cdot\vec{B})=0$ ***so the particle will not deflect after passing through the uniform magnetic field. Hence we must use a non-uniform magnetic field in the Stern-Gerlach experiment.***

21. For a spin-1/2 particle, e.g., a silver atom in its ground state, the initial spin wave function of the particle $\Psi(0)=a\left|\uparrow\right\rangle_z+b\left|\downarrow\right\rangle_z$ gets spatially separated into two parts after passing through a SGZ. If we choose an atom with the total spin quantum number $s=1$, the wave function will spatially separate into at most how many parts, depending upon the initial state, after passing through an SGZ?

A. 2
B. 3
C. 4
D. 5

22. Suppose we did not know about spin angular momentum. We choose silver atoms with the total orbital angular momentum quantum number $\ell=0$ and send them through the SGZ. We expect the wave function to spatially separate into at most how many parts, depending on the initial wave function associated with L?

A. 1
B. 2
C. 3
D. 4

→[explanation] ***If we don't know about spin, we would expect to see only one cluster of spots (one spot corresponds to one impact) on the screen for the silver atoms. The number of cluster of spots you observe depends on the number of discrete components of total angular momentum*** $\vec{J}$***, where*** $\vec{J} = \vec{L} + \vec{S}$***. If the spin quantum number is zero, then*** $\vec{J} = \vec{L}$ ***and all the splitting involved by passing through the SGA will be determined by the orbital quantum number*** ℓ***. But if*** $\ell = 0$***, then*** $\vec{J} = \vec{S}$ ***and the splitting involved by passing through the SGA will be determined by the spin quantum number*** s***. Actually, the Stern-Gerlach experiment was first conducted in 1922. At that time, the concept of spin was not known. So you can imagine the confusion when two clusters of spots were observed with neutral silver atoms passing through the SGA.***

25. Consider the following conversation between Pria and Mira:

Pria: The eigenstates of $\hat{S}_x$ are orthogonal to the eigenstates of $\hat{S}_z$.

Mira: I disagree. However, the different eigenstates of $\hat{S}_x$ are orthogonal to one another. In fact, the orthogonal eigenstates of $\hat{S}_x$ form one basis for the vector space for the spin degrees of freedom and the orthogonal eigenstates of $\hat{S}_z$ form another basis.

With whom, if either, do you agree? Explain.
(a) Pria
(b) Mira
(c) Neither

→[explaination] ***For spin-1/2 particle, spin space is 2-D, not 3-D. In the*** $\hat{S}_z$ ***basis, the two basis vectors are*** $\left|\uparrow\right\rangle_z$ ***and*** $\left|\downarrow\right\rangle_z$ ***which are orthogonal to each other (*** ${}_z\left\langle\uparrow\middle|\downarrow\right\rangle_z = 0$ ***). If we change the basis to eigenstates of*** S_x***, the basis vectors will be*** $\left|\uparrow\right\rangle_x$ ***and*** $\left|\downarrow\right\rangle_x$ ***which are also orthogonal to each other. But different eigenstates of*** $\hat{S}_x$ ***and*** $\hat{S}_z$ ***are not orthogonal to each other, e.g.,*** ${}_z\left\langle\uparrow\middle|\downarrow\right\rangle_x \neq 0$***. We can draw an analogy with a map in which we can choose north and east directions as orthogonal basis vectors, or we can choose southeast and northeast as another orthogonal basis vectors. But north and northeast are not orthogonal to each other in the map.***

26. Consider the following conversation between Pria and Mira:

Pria: If an atom in the state $|\uparrow\rangle_z$ were sent through an SGX-, the atom will be unaffected by the field gradient. The atom cannot feel a force because its state $|\uparrow\rangle_z$ is perpendicular to the magnetic field.

Mira: I disagree. You are getting confused between a state in the Hilbert space and the magnetic field in the physical space. We can write $|\uparrow\rangle_z$ in terms of the eigenstates of $\hat{S}_x$ and the magnetic field gradient in the *x*-direction will spatially separate the $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$ components.

With whom, if either, do you agree? Explain.

27. To analyze the wavefunction after an atom in an initial state $|\uparrow\rangle_z$ passes through SGX-, which basis is most appropriate, eigenstates of $\hat{S}_x$ or $\hat{S}_z$? Explain. (Hint: As we discussed earlier, in the classical analogy, the force on the particles will make the particles with different eigenstates of $\hat{S}_x$ spatially separated when passing through an SGX-. Also, note that the Hamiltonian commutes with $\hat{S}_x$ for SGX so that eigenstates of $\hat{S}_x$ are the energy eigenstates.)

28. Is the magnetic field a vector field in a 3D physical space (in which the experiment is carried out) or a 2D Hilbert space (in which the two spin states of the system lie)? Explain.

29. Is the initial state of the system $|\uparrow\rangle_z$ a vector in a 3D physical space or a 2D Hilbert space? Explain.

30. Is it appropriate to say that a vector in a 2D Hilbert space is orthogonal to a vector in a 3D physical space? Explain. (Note : The vector $\vec{B}$ in a 3D physical space has 3 components and a vector in a 2D Hilbert space has only 2 components.)