Larmor Precession of Spin

Notation: $\hat{S}_z|\uparrow\rangle_z = \frac{\hbar}{2}|\uparrow\rangle_z$, *and* $\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar}{2}|\downarrow\rangle_z$.

For reference, the eigenstates of $\hat{S}_x$ *and* $\hat{S}_y$ *are given by:*

$|\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$, $|\downarrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z - |\downarrow\rangle_z)$

$|\uparrow\rangle_y = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + i|\downarrow\rangle_z)$, $|\downarrow\rangle_y = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z - i|\downarrow\rangle_z)$

Initial state $|\uparrow\rangle_z$*, measure* S_z *(Questions 1-7)*

1. The Hamiltonian of a spin-1/2 particle in a magnetic field $\vec{B} = B_0\hat{z}$ is $\hat{H} = -\gamma B_0 \hat{S}_z$, where $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the z-component of the spin angular momentum operator. Which one of the following equations correctly represents the eigenvalue of the Hamiltonian $\hat{H} = -\gamma B_0 \hat{S}_z$?

A. $E_\pm = \mp\gamma B_0$

B. $E_\pm = \mp\frac{\gamma B_0}{2}$

C. $E_\pm = \mp\frac{\hbar}{2}\gamma B_0$

D. $E_\pm = \mp\frac{\hbar}{2}$

2. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = |\uparrow\rangle_z$. Which one of the following equations correctly represents the state $|\chi(t)\rangle$ of the electron? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $|\chi(t)\rangle = |\uparrow\rangle_z$

B. $|\chi(t)\rangle = e^{i\gamma B_0 t/2}|\uparrow\rangle_z$

C. $|\chi(t)\rangle = e^{i\gamma B_0 t/2}|\uparrow\rangle_z + e^{-i\gamma B_0 t/2}|\downarrow\rangle_z$

D. $|\chi(t)\rangle = ae^{i\gamma B_0 t/2}|\uparrow\rangle_z + be^{-i\gamma B_0 t/2}|\downarrow\rangle_z$

3. Consider the following conversation between Student 1 and Student 2 about the above Hamiltonian operator.

Student 1: The eigenstates of $\hat{S}_z$ will also be the eigenstates of $\hat{H}$ because $\hat{H}$ is proportional to $\hat{S}_z$ except for some multiplicative constants.

Student 2: No. The presence of the magnetic field will make the eigenstates of $\hat{S}_z$ and $\hat{H}$ different. The eigenstates of $\hat{H}$ will change with time in a non-trivial manner.

Student 1: I disagree. If the magnetic field had a time dependence, e.g., $B = B_0 \cos(\omega t)\hat{k}$, the eigenstates of $\hat{H}$ will change with time in a non-trivial manner but not for the present case where $\vec{B}$ is constant.

With whom do you agree? Explain your reasoning.

A. Student 1

B. Student 2

C. Neither

4. In question 2, suppose $t = t_0$ satisfies $-\gamma B_0 t_0/2 = \pi/6$. What is the probability of obtaining $|\uparrow\rangle_z$ when we measure S_z at $t = t_0$?

Now we will use simulations to verify our results. In the simulations, the red block is the strong uniform magnetic field that may cause Larmor precession and is central to this lesson on Larmor precession. The blue block is the Stern-Gerlach apparatus which is not part of the setup for Larmor precession but is being used to detect the spin state of the system after going through the strong uniform magnetic field. For example, when the upper detector after SGZ clicks, the state of the particle measured is $|\uparrow\rangle_z$. The number "30" in the magnetic field means that the particle will stay in the magnetic field for time t_0 satisfying $-\gamma B_0 t_0/2 = \pi/6$ $(= 30°)$.

5. Now you can use the simulation ($|\chi(0)\rangle = |\uparrow\rangle_z$, γ B t_0 / 2 = 30° , Measure S_z) to check your answer to the previous problem (*question 4*). You can click the button "Step 1000" to send 1000 atoms through the uniform magnetic field followed by a SGZ. The probability of obtaining $|\uparrow\rangle_z$ can be estimated through the counts at the detectors placed after SGZ. Explain whether what you observed in the simulation is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

6. In *question 4*, does the probability of getting $|\uparrow\rangle_z$ depend on time t? Explain.

Now you can use the other simulations with $|\chi(0)\rangle = |\uparrow\rangle_z$ that involve measuring S_z to check your answer. Does the probability of the up detector clicking (probability of getting $|\uparrow\rangle_z$) differ in these cases? In these three simulations, the time t_0 satisfies $-\gamma B_0 t_0/2 = \pi/6$ $(= 30°)$, $-\gamma B_0 t_0/2 = \pi/4$ $(= 45°)$ and $-\gamma B_0 t_0/2 = \pi/2$ $(= 90°)$ respectively.

7. Explain whether what you observed in the simulations above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

Initial state $|\uparrow\rangle_z$, measure S_x (Questions 8-15)

8. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = |\uparrow\rangle_z$. After time t, the electron will be in state $|\chi(t)\rangle$. Write down $|\chi(t)\rangle$ in the basis of eigenstates of $\hat{S}_x$, i.e., $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$.

9. In the previous problem (*problem 8*), suppose at time $t = t_0$, $-\gamma B_0 t_0/2 = \pi/6$. What is the probability of getting $|\uparrow\rangle_x$ when we measure the observable S_x at $t = t_0$?

Now you can use the simulation with $|\chi(0)\rangle = |\uparrow\rangle_z$ that involves measuring S_x to check your answer. You can click the button "Step 1000" to send 1000 atoms through the magnetic field and SGX. So the probability of getting $|\uparrow\rangle_x$ can be estimated through the counts in the detectors placed after SGX.

10. Explain whether what you observed in the simulation above is consistent with your prediction in question *9*. If it is not consistent, resolve this discrepancy.

11. In *question 9*, does the probability of obtaining $|\uparrow\rangle_x$ depend on the time t?

Now you can the other simulations with $|\chi(0)\rangle = |\uparrow\rangle_z$ that involve measuring S_x to check your answer. Does the probability of the up detector clicking (probability of getting $|\uparrow\rangle_x$) differ in these cases?

12. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

13. Calculate the expectation value of S_x at the time t when the initial state at $t=0$ was $|\uparrow\rangle_z$. Then evaluate $\langle S_x\rangle$ at time $\boldsymbol{t_0}$ satisfying $\boldsymbol{-\gamma B_0 t_0/2 = \pi/6}$ $(= \mathbf{30°})$.

14. Send 1000 atoms through the magnetic field in the simulation with $|\chi(0)\rangle = |\uparrow\rangle_z$, $\boldsymbol{-\gamma B_0 t_0/2 = \pi/6}$ $(= \mathbf{30°})$ that measures S_x. Approximately find the expectation value $\langle S_x\rangle$ by using the number of clicks in the upper and lower detectors in your calculation. Is this result approximately the same as what you calculated in the previous question (*problem 13*)?

15. If you had to explain to a friend what expectation value physically means and why the calculated value and the value from the simulation are approximately the same, what would you say?

Initial state $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, $\hat{H} = -\gamma B_0 \hat{S}_z$, *measure* S_z *(Questions 16-25)*

16. If the state of the system at time *t*=0 is given by $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, which one of the following equations correctly represents the state $|\chi(t)\rangle$ after time *t*? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $|\chi(t)\rangle = e^{i\gamma B_0 t/2}(a|\uparrow\rangle_z + b|\downarrow\rangle_z)$
B. $|\chi(t)\rangle = e^{-i\gamma B_0 t/2}(a|\uparrow\rangle_z + b|\downarrow\rangle_z)$
C. $|\chi(t)\rangle = e^{i\gamma B_0 t/2}((a+b)|\uparrow\rangle_z + (a-b)|\downarrow\rangle_z)$
D. $|\chi(t)\rangle = a e^{i\gamma B_0 t/2}|\uparrow\rangle_z + b e^{-i\gamma B_0 t/2}|\downarrow\rangle_z$

17. In the previous problem (*question 16*), what is the probability of obtaining $\pm\hbar$ if we measure S_z at time $t = t_0$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

A. $\hbar/2$ with a probability $a e^{i\gamma B_0 t/2}$ and $-\hbar/2$ with a probability $b e^{-i\gamma B_0 t/2}$
B. $\hbar/2$ with a probability $a^2 e^{i\gamma B_0 t/2}$ and $-\hbar/2$ with a probability $b^2 e^{-i\gamma B_0 t/2}$
C. $\hbar/2$ with a probability $|a^2|$ and $-\hbar/2$ with a probability $|b^2|$
D. $\hbar/2$ and $-\hbar/2$ with equal probability

18. Consider the following conversation between Student 1 and Student 2 about measuring S_z in the state $|\chi(t)\rangle$

Student 1: Since the probability of measuring $\hbar/2$ is $|a^2|$, that of measuring $-\hbar/2$ is $|b^2|$ and $|a^2| + |b^2| = 1$, we can choose our a and b as $a = e^{i\varphi_1} cos(\alpha)$ and $b = e^{i\varphi_2} sin(\alpha)$ where α, φ_1 and φ_2 are free parameters.

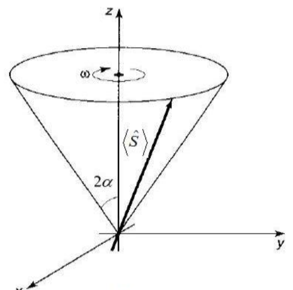
Student 2: I agree. Since $cos^2(\alpha) + sin^2(\alpha) = 1$, it gives the same relation as $|a^2| + |b^2| = 1$ and there is no loss of generality.

Do you agree with Student 1 and Student 2? If yes, for the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$, what are the values of a and b when you express it in terms of $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$? And what are the values of α, φ_1 and φ_2 when you express it in terms of α defined earlier as $a = e^{i\varphi_1} cos(\alpha)$ and $b = e^{i\varphi_2} sin(\alpha)$.

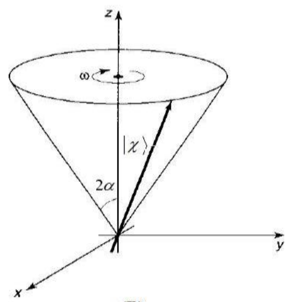
19. Suppose the initial state is $|\chi(0)\rangle = cos\,\alpha\,|\uparrow\rangle_z + sin\,\alpha\,|\downarrow\rangle_z$. Which one of the following is the expectation value $\langle S_z\rangle = \langle\chi(t)|\hat{S}_z|\chi(t)\rangle$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$. $sin\,2\,\alpha = 2\,sin\,\alpha\,cos\,\alpha$, $cos\,2\,\alpha = cos^2\,\alpha - sin^2\,\alpha = 1 - 2\,sin^2\,\alpha = 2\,cos^2\,\alpha - 1$.

A. $cos(2\alpha)\,cos(\gamma B_0 t)\hbar/2$
B. $sin(2\alpha)\,sin(\gamma B_0 t)\hbar/2$
C. $cos(2\alpha)\hbar/2$
D. $sin(2\alpha)\hbar/2$

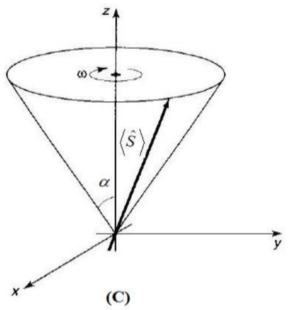
20. Which one of the following illustrations correctly shows the relationship between $\langle\vec{S}\rangle$ or $|\chi\rangle$ and the angle α?



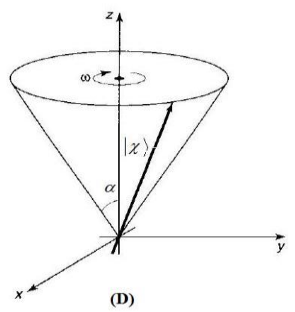
(A)



(B)



(C)



(D)

21. Consider the following conversation between Student 1 and Student 2 about the previous problem (question 20)

Student 2: I don't understand why choice D in the previous problem is wrong. Isn't $|\chi\rangle$ a vector that is precessing? In fact, the initial state of the system is $|\chi(0)\rangle = cos\, \alpha\, |\uparrow\rangle_z + sin\, \alpha\, |\downarrow\rangle_z$.

Student 1: Well, the spin state $|\chi\rangle$ is a vector in the 2-D Hilbert space instead of 3-D physical space. So we cannot show the vector $|\chi\rangle$ in the x-y-z coordinate system.

Do you agree or disagree with Student 1? Explain.

22. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$. Write down $|\chi(t)\rangle$ explicitly. Suppose $t = t_0$ satisfies $-\gamma B_0 t_0/2 = \pi/6$. What is the probability of getting $|\uparrow\rangle_z$ when we measure the observable S_z at $t = t_0$? The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

Now you can use the simulation with $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z) = |\uparrow\rangle_x$, that has $-\gamma B_0 t_0/2 = \pi/6$, and involves measuring S_z to check your answer. You can click the button "Step 1000" to send 1000 atoms through the magnetic field and SGZ. So the probability of getting $|\uparrow\rangle_z$ can be estimated through the counts in the detectors placed after SGZ.

23. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question. If it is not consistent, resolve this discrepancy.

24. In *question 22*, does the probability of getting $|\uparrow\rangle_z$ depend on the time t? What is the expectation value of S_z at time t? Does $\langle S_z\rangle$ depend on time?

Now use the other simulations with initial state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z) = |\uparrow\rangle_x$ that involve measuring S_z to check your answer to question 24. Does the probability of the up detector clicking (probability of obtaining $|\uparrow\rangle_z$) differ in these cases? The time t_0 satisfies $-\gamma B_0 t_0/2 = \pi/6$ $(= 30^\circ)$, $-\gamma B_0 t_0/2 = \pi/4$ $(= 45^\circ)$ and $-\gamma B_0 t_0/2 = \pi/2$ $(= 90^\circ)$ in these three simulations respectively.

25. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question (*question 24*). If it is not consistent, resolve this discrepancy.

Initial state $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, *measure* S_x *(Questions 26-32)*

Now let's consider a general initial state $|\boldsymbol{\chi(0)}\rangle = \boldsymbol{a}|\uparrow\rangle_{\boldsymbol{z}} + \boldsymbol{b}|\downarrow\rangle_{\boldsymbol{z}}$ ***at time t=0. The Hamiltonian is*** $\widehat{\boldsymbol{H}} = -\boldsymbol{\gamma B_0 \widehat{S}_z}$***. Answer problems 26 and 28.***

26. Find the state $|\chi(t)\rangle$ after time *t*. Express your answer in terms of α defined earlier as $a = e^{i\varphi_1} cos(\alpha)$ and $b = e^{i\varphi_2} cos(\alpha)$.

27. Suppose the initial state is $|\chi(0)\rangle = cos\,\alpha\,|\uparrow\rangle_z + sin\,\alpha\,|\downarrow\rangle_z$. Which one of the following is the expectation value $\langle S_x\rangle = \langle\chi(t)|\hat{S}_x|\chi(t)\rangle$? The Hamiltonian operator is $\widehat{H} = -\gamma B_0 \hat{S}_z$.

A. $cos(2\alpha)\,cos(\gamma B_0 t)\hbar/2$

B. $sin(2\alpha)\,cos(\gamma B_0 t)\hbar/2$

C. $cos(2\alpha)\hbar/2$

D. $sin(2\alpha)\hbar/2$

28. Consider the following conversation between Student 1 and Student 2.

Student 2: I remember that in *question 20*, $\langle\vec{S}\rangle$ rotates about the z-axis with an angular frequency ω. So $\langle S_x\rangle$ must depend on time.

Student 1: Yes, you are right. $\langle S_x\rangle$ is the projection of $\langle\vec{S}\rangle$ on the x-axis. Since $\langle S_x\rangle = sin(2\alpha)\,cos(\gamma B_0 t)\hbar/2$, we know the angular frequency is $\omega = \gamma B_0$.

Do you agree with Student 2 and Student 1? Explain.

29. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$. The Hamiltonian operator is $\widehat{H} = -\gamma B_0 \hat{S}_z$. Write $|\chi(t)\rangle$ in the most appropriate basis to find the probability of getting $|\uparrow\rangle_x$ if you measure S_x at time *t*. Does the probability of obtaining $|\uparrow\rangle_x$ depend on time t? Explain.

Now use all four of the simulations that have initial state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z) = |\uparrow\rangle_x$ **that involve measuring S_x to check your answer. Does the probability of the up detector clicking (probability of obtaining $|\uparrow\rangle_x$) differ in these cases? The times $\boldsymbol{t_0}$ satisfy t_0=0, $-\boldsymbol{\gamma B_0 t_0/2 = \pi/6}$ $(= \mathbf{30}°)$, $-\boldsymbol{\gamma B_0 t_0/2 = \pi/4}$ $(= \mathbf{45}°)$ and $-\boldsymbol{\gamma B_0 t_0/2 = \pi/2}$ $(= \mathbf{90}°)$ in these four simulations.**

30. In the previous problem (*question 29*), the initial state is just $|\chi(0)\rangle = |\uparrow\rangle_x$. Why then does the probability of getting $|\uparrow\rangle_x$ depend on time? Explain. (Hint: Can we write the state $|\chi(t)\rangle$ at time t as $|\chi(t)\rangle = e^{i\gamma B_0 t/2}|\uparrow\rangle_x$?)

31. In *problem 29*, what is the expectation value of S_x at time t? Does $\langle S_x \rangle$ depend on time? If $t = t_0$ satisfies $-\gamma B_0 t_0/2 = \pi/6$, what is $\langle S_x \rangle$ at $t = t_0$?

Now you can use the simulation with $|\chi(0)\rangle = |\uparrow\rangle_x$ $\boldsymbol{-\gamma B_0 t_0/2 = \pi/6}$ $(= \mathbf{30}°)$ that involves measuring S_x to check your answer. You can click the button "Step 1000" to send 1000 atoms through the uniform magnetic field and SGX. The probabilities of getting $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$ can be estimated through the counts in the detectors placed after SGX. You can use these probabilities to calculate $\langle S_x \rangle$.

32. Explain whether what you observed in the simulation above is consistent with your prediction in *question 31*.

Initial state $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, *measure* S_y. *(Questions 33-37)*

33. An electron in a magnetic field $\vec{B} = B_0\hat{z}$ is initially in the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$. The Hamiltonian operator is $\widehat{H} = -\gamma B_0 \hat{S}_z$. Write down $|\chi(t)\rangle$ explicitly. What is the probability of getting $|\uparrow\rangle_y$ if we measure $\mathbf{S_y}$ at $t = t_2$? t_2 satisfies $-\gamma B_0 t_2/2 = \pi/4$.

Now use the simulation with $|\chi(0)\rangle = |\uparrow\rangle_x$, $\boldsymbol{-\gamma B_0 t_0/2 = \pi/6}$ $(= \mathbf{30}°)$ that involves measuring S_y to check your answer. You can click the button "Step 1000" to send 1000 atoms through the magnetic field and SGY. The probability of getting $|\uparrow\rangle_y$ and $|\downarrow\rangle_y$ can be estimated through the counts in the detectors placed after SGY.

34. Explain whether what you observed in the simulation above is consistent with your prediction in the previous question.

35. Explain in your own words why $\langle S_z \rangle$ above does not depend on time whereas $\langle S_x \rangle$ and $\langle S_y \rangle$ do. The Hamiltonian operator is $\widehat{H} = -\gamma B_0 \hat{S}_z$.

36. The time-dependence of expectation value of any observable A whose corresponding operator is $\hat{A}$ is given by $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle + \left\langle\frac{\partial \hat{A}}{\partial t}\right\rangle$. Check whether your answers for $\langle S_x\rangle$ (*question 27*) and $\langle S_z\rangle$ (*question 19*) are consistent with this equation.

37. Consider the following conversation between Student 3 and Student 4 about $\langle S_z\rangle$ in an arbitrary state $|\chi(t)\rangle$:

Student 3: If the state of the system $|\chi(t)\rangle$ evolves in time, the expectation value $\langle S_z\rangle$ will depend on time.

Student 4: I disagree with the second part of your statement. The time development of the expectation value of any observable A whose corresponding operator is $\hat{A}$ is given by $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle + \left\langle\frac{\partial \hat{A}}{\partial t}\right\rangle$. In our case, $[\hat{H}, \hat{S}_z] = 0$ and the operator $\hat{S}_z$ does not have any explicit time dependence so $\frac{\partial \hat{S}_z}{\partial t} = 0$. Thus, $\frac{d\langle S_z\rangle}{dt} = 0$ and the expectation value will not change with time.

With whom do you agree? Explain.

The general rules for the time-dependence of expectation value in Larmor precession (38 - 48)

In problems 38 - 39, the Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

38. Choose all of the following statements that are true about the expectation value $\langle\vec{S}\rangle$ in the state $|\chi(t)\rangle$:

A. $\langle\vec{S}\rangle$ must always depend on time because $[\hat{H}, \hat{S}] \neq 0$

B. $\langle\vec{S}\rangle$ is time-independent because the expectation value of an observable is its time-averaged value.

C. $\langle\vec{S}\rangle$ is time-independent only when the initial state is purely $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$.

39. Choose all of the following statements that are true about the expectation value $\langle\vec{S}\rangle$ in the state $|\chi(t)\rangle$ when the initial state is **NOT** purely $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$:

(1) The z component of $\langle\vec{S}\rangle$, i.e., $\langle S_z\rangle$, is time independent.

(2) The x and y components of $\langle\vec{S}\rangle$ change with time. When $\langle S_x\rangle$ is maximum $\langle S_y\rangle$ is a minimum and vice versa.

(3) The magnitudes of the maximum values of $\langle S_x\rangle$ and $\langle S_y\rangle$ are the same.

A. 1 and 2

B. 1 and 3

C. 2 and 3

D. All of the above

40. Consider a classical particle in a 3-D space. If the acceleration of the particle is always perpendicular to the velocity of the particle, will the magnitude of the velocity change with time? Will the direction of the velocity change with time? Explain.

41. Choose all of the following statements that are correct for the angular momentum $\vec{L}$ of a classical particle which is precessing.
    (1) In classical mechanics, the rate of changing of the angular momentum $\vec{L}$ is determined by $\frac{d\vec{L}}{dt} = \vec{\tau}$, where $\vec{\tau}$ is the external torque.
    (2) If the external torque $\vec{\tau}$ is always perpendicular to $\vec{L}$ and the magnitude of $\vec{\tau}$ is a constant, it will cause a change in the direction of $\vec{L}$ but not a change in the magnitude of $\vec{L}$.
    (3) If the external torque $\vec{\tau}$ is always perpendicular to $\vec{L}$ and the magnitude of $\vec{\tau}$ is a constant, $\vec{L}$ will precess with a constant angular speed ω.

A. only 1
B. 1 and 2 only
C. 1 and 3 only
D. 2 and 3 only
E. All of the above.

42. According to the Ehrenfest theorem, expectation values of physical observables follow classical laws. Using the analogy with the classical precession, write an expression for the precession of the expectation value of the spin angular momentum $\langle \vec{S} \rangle$ in terms of the torque due to an external magnetic field. Remember that the torque for a particle in an external magnetic field $\vec{B}$ is $\vec{\tau} = \vec{\mu} \times \vec{B} = \gamma \vec{S} \times \vec{B}$, where $\vec{\mu}$ is the magnetic dipole moment.

In problems 43 - 46, the Hamiltonian operator is $\boldsymbol{\hat{H} = -\gamma B_0 \hat{S}_z}$.

43. Choose all of the following statements that are true about the expectation value $\langle \vec{S} \rangle$ in the state $|\chi(t)\rangle$ when the initial state is **NOT** purely $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$. The external magnetic field is in the *z*-direction.
    (1) The vector $\langle \vec{S} \rangle$ can be thought to be precessing about the z axis at an angle 2α.
    (2) The vector $\langle \vec{S} \rangle$ can be thought to be precessing about the z axis with a frequency $\omega = \gamma B_0$.
    (3) All the three components of vector $\langle \vec{S} \rangle$ change as it precesses about the z axis.

A. 1 and 2 only
B. 1 and 3 only
C. 2 and 3 only
D. All of the above

44. Choose a three-dimensional coordinate system in the spin space with the z axis in the vertical direction. Draw a sketch showing the precession of $\langle \vec{S} \rangle$ about the z axis when the state of the system starts out in $|\chi(0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$.

45. In the previous problem (*question 44*), show the projection of $\langle \vec{S} \rangle$ along the x, y and z axes at two separate times. Explain in words why the projection of $\langle \vec{S} \rangle$ along the z direction does not change with time but those along the x and y directions change with time.

46. Explain in your own words why, for $\hat{H} = -\gamma B_0 \hat{S}_z$, if the initial state is $|\uparrow\rangle_z$, $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$ are all time independent. However, if the initial state is $\frac{1}{\sqrt{2}}(|\uparrow\rangle_z + |\downarrow\rangle_z)$, only $\langle S_z \rangle$ is time independent.

47. If a quantum system is in an eigenstate $|q\rangle$ of an operator $\hat{Q}$ corresponding to an observable Q, will the expectation value of Q depend on time? Explain your reasoning.

48. If the operator $\hat{Q}$ commutes with the Hamiltonian $\hat{H}$, will the value of $\frac{d}{dt}\langle \psi | \hat{Q} | \psi \rangle$ depend on the wavefunction $\psi(x,t)$? Explain your reasoning.