**Dirac Notation Homework**

* Throughout this homework, the normalization issues for the position and momentum eigenstates are ignored.
* In all of the questions below, denotes a generic state of a quantum system with Hamiltonian .
* Assume that the Hilbert space is infinite dimensional.
* For all questions involving a generic operator , assume that it only depends on position and momentum and has no explicit time dependence.
* Assume the particle is confined in a one dimensional physical “laboratory” space.
* Assume refers to a summation over a complete set of states ().

**The goals of this homework are to help you learn about:**

* **Connecting Dirac notation with function space (position and momentum representation)**
* **Relationship between state vector and the wavefunction** 
  + In the position representation, is the projection of state vector along the eigenstate of position (position eigenstates {} form a complete set of basis vectors). The column vector (considered as a function of ) is called the position space wavefunction.
  + In the momentum representation, is the projection of state vector along the eigenstate of momentum (momentum eigenstates {} form a complete set of basis vectors). The column vector (considered as a function of ) is called the momentum space wavefunction.
  + The position space wavefunction and momentum space wavefunction can be considered as infinite dimensional column vectors.
  + The momentum space wavefunction and position space wavefunction are a Fourier transform and inverse Fourier transform of each other, respectively.
* **Position and momentum eigenstates**
  + The position eigenstate with eigenvalue can be written as:
    - in Dirac notation.
    - in position representation. It is also called the position eigenfunction in position representation and a form of the orthogonality relation.
    - in the momentum representation. It is also called the position eigenfunction in momentum representation.
  + The momentum eigenstate with eigenvalue can be written as:
    - in Dirac notation
    - in the position representation. It is also called the momentum eigenfunction in position representation.
    - in the momentum representation. It is also called the momentum eigenfunction in momentum representation and a form of the orthogonality relation.
* **Position and momentum representation**
  + For a generic state and a generic operator (which depends only on operators and ):
    - In the **position** representation, is represented by
    - In the **momentum** representation, is represented by
* **For a given quantum system, the Hamiltonian operator and its eigenstates (energy eigenstates)**
  + for , represent a complete set of energy eigenstates with eigenvalue .
  + The eigenvalue equation for the Hamiltonian operator is .
  + The expectation value of energy in state  is.

**Connecting Dirac notation with function space (position and momentum representation):**

* **Relationship between state vector and the wavefunction for a given quantum system**

1. Choose all of the following statements that are correct about the generic state vector
2. is a representation of the state vector in the position representation and is called the position space wavefunction.
3. is a representation of the state vector in the momentum representation and is called the momentum space wavefunction.
4. The state vector can be written as a column vector once a basis has been chosen.
5. (I) only
6. (I) and (II) only
7. (I) and (III) only
8. (II) and (III) only
9. All of the above.
10. Consider the following conversation between two students:

* Student A: In the preceding question, we can also write as .
* Student B: I disagree. , where the sign means that the equality is valid only with respect to a chosen basis. is a representation of in position representation when we choose a complete set of position eigenstates, , as our basis vectors, so we can write

With whom do you agree? Explain your reasoning.

1. Earlier you learned that in a three dimensional space, a generic state can be written as , where form an orthonormal basis and , , and are complex numbers. Thus, a generic state can be written as a column vector once a basis is chosen, i.e.,

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where the sign means that the equality is valid only with respect to a chosen basis.

In quantum mechanics, the generic state vector is a vector in the Hilbert space, which is infinite dimensional. The generic state can be written in terms of a linear superposition of a complete set of eigenstates of any hermitian operator .

1. Suppose a hermitian operator has a complete set of eigenstates {, } with discrete eigenvalues . Write in terms of a linear superposition of a complete set of eigenstates {}.
2. Keeping in mind how the generic state is written as a column vector, write as an infinite dimensional column vector with respect to the orthonormal basis .
3. Consider a generic state vector , where . Then, the state vector can be represented as a column vector like this: , where the sign means that this is a representation of in the chosen basis, e.g., {, }. Consider the following conversation between two students about the situation where basis vectors are chosen to be position eigenstates or momentum eigenstates , each of which have a continuous eigenvalue spectrum.

* Student 1: We cannot write state vector as a column vector if position eigenstates are chosen as the basis vectors. State vector written as a linear superposition of position eigenstates is , where . But since this expansion of involves an integral instead of a summation, we cannot write as a column vector with respect to the basis vectors .
* Student 2: I disagree. Even though the expansion of is an integral instead of a summation, we can still envision as a column vector with respect to the basis vectors . Like this:

.

* Student 1: But why do the ’s have indices? Don’t position eigenstates have a continuous eigenvalue spectrum , not a discrete eigenvalue spectrum?
* Student 2: Yes, you are correct. Actually, you should think of , … etc., and take the limit as . I was simply making an analogy with the discrete eigenvalue spectrum case. However, the best way to write when position eigenstates are chosen as the basis vectors is as , which is also called the position space wavefunction. is a column vector with position eigenvalues as a continuous index.

Do you agree with Student 2? Explain your reasoning.

1. Consider the following conversation between three students:

* Student A: How does this expansion of in terms of a complete set of position eigenstates , , help you in questions involving the measurement of the position of the particle?
* Student B: gives us the probability of finding the particle in a narrow range between and when we measure the position of the particle.
* Student C: But I thought that the probability of finding the particle in a narrow range between and was .
* Student A: So is the probability of finding the particle in a narrow range between and represented mathematically as ?
* Student B: No. It is just not . is the probability density at position . We multiply by a width to obtain the probability of finding the particle in a narrow range between and .

Do you agree with Student B’s explanation? Explain your reasoning.

1. Write the generic state vector as a column vector in the momentum representation when momentum eigenstates are chosen as basis vectors.
2. Write the probability of finding the particle with a momentum between and when we measure the momentum of the particle.

**Position and Momentum Eigenstates**

* **Position eigenstates**

1. Choose all of the following equations that are correct for the position eigenstate with eigenvalue .
2. (I) and (II) only
3. (I) and (III) only
4. (II) and (III) only
5. All of the above.
6. Consider the following conversation between Student A and Student B.

* Student B**:** I don’t understand how and can both be correct. They are two different eigenvalue equations for the same operator , so shouldn’t one of them be incorrect?
* Student A**:** Actually, and convey the same information. The former is the eigenvalue equation for the position operator in Dirac notation, and the latter is the eigenvalue equation for the position operator in the position representation. In the position representation, is equivalent to a multiplication by . We can also write since , which is zero for all position eigenvalues except when .

With whom do you agree? Explain.

1. A) Which one of the following is the eigenstate of position with eigenvalue written in position representation, i.e., ?
2. B) The position eigenstate written in position representation (when considered as a function of ) is called the:
3. Position eigenfunction in position representation.
4. Position eigenfunction in momentum representation.
5. Position eigenfunction either in position or momentum representation since the expression for position eigenfunction is the same regardless of the representation.
6. None of the above.
7. Consider the following conversation between two students:

* Student 1: The position eigenfunction should always be a delta function whether we write it in position or momentum representation.
* Student 2: I disagree. When position eigenstate with eigenvalue is written by choosing position eigenstates as basis vectors, we obtain which is the position eigenfunction in the position representation. When the position eigenstate is written by choosing momentum eigenstates as basis vectors, we obtain , which is the position eigenfunction in the momentum representation. However, is not a delta function. The position eigenfunction is only a delta function in the position representation, but not in the momentum representation.

With whom do you agree? Explain your reasoning.

1. Which one of the following is the eigenstate of position with eigenvalue written in momentum representation, i.e., (position eigenfunction in momentum representation)?

* **Momentum eigenstates**

1. Choose all of the following equations that are correct for the momentum eigenstate with eigenvalue .
2. (I) and (II) only
3. (I) and (III) only
4. (II) and (III) only
5. All of the above.
6. Which one of the following is the eigenstate of momentum with eigenvalue in momentum representation, i.e., (momentum eigenfunction in momentum representation)?
7. Which one of the following is the eigenstate of momentum in position representation, i.e., (momentum eigenfunction in position representation)?
8. Substitute the expression you chose for the momentum eigenfunction in the position representation in the preceding question and the expression for the momentum operator (one dimensional) in position representation, , in the eigenvalue equation for momentum, , where is the eigenstate of momentum with eigenvalue in position representation Check whether the eigenvalue equation is satisfied. If not, correct your answer to the preceding question. Show **all** your work in the space provided below.
9. The eigenstates of a hermitian operator with different eigenvalues are orthonormal[[1]](#footnote-1) to each other. Position and momentum operators ( and ) each have a continuous eigenvalue spectrum. Assume that the Hamiltonian operator () for a given quantum system has a discrete eigenvalue spectrum. Choose all of the following that are correct about the scalar product of two eigenstates of an operator.
   1. where E denotes energy.

*Note: In more common notation*   *can be written as* , *where* .

1. (III) only
2. (I) and (II) only
3. (I) and (III) only
4. All of the above.
5. Consider the following conversation between two students about a position eigenfunction and a momentum eigenfunction in the position representation:

* Student 1: A position eigenstate is represented in position representation as .

is a special type of position space wavefunction in which position has a definite value of . Before taking the limit to obtain the delta function, a position eigenfunction in the position representation, , is a very sharply peaked wavefunction about in position representation.

* Student 2: I agree with you. In addition, a momentum eigenstate is represented in position representation as . is another special type of position space wavefunction where momentum has a definite value . Instead of being sharply peaked like a delta function, a momentum eigenfunction in the position representation is spread out as (which is a linear combination of sine and cosine functions over all position with a definite momentum and wave number ) and the probability density is uniform.

Do you agree with Student 1, Student 2, both, or neither? Explain your reasoning.

1. Consider the following conversation between two students:

* Student 1: is a special type of momentum space wavefunction , in which momentum has a definite value . A momentum eigenfunction in the momentum representation, , is a very sharply peaked wavefunction about in momentum representation.
* Student 2: I agree with you. In addition, a position eigenstate with eigenvalue represented in momentum representation is . is also a special type of momentum space wavefunction in which position has a definite value ( is fixed in , which is a function of ).

Do you agree with Student 1, Student 2, neither, or both? Explain your reasoning.

1. Is the position eigenstate (or position eigenfunction) in the momentum representation that you learned about in the preceding question localized or an extended function of momentum (consider the real and imaginary parts)? Explain your reasoning.
2. You have learned about position and momentum eigenstates in the position representation and momentum representation. Using what you have learned, fill out the table below:

|  |  |  |
| --- | --- | --- |
|  | Position representation | Momentum representation |
| Position eigenstates  (or position eigenfunctions) |  |  |
| Momentum eigenstates  (or momentum eigenfunctions) |  |  |

* **Position and Momentum Operators**

Up to this point, you have learned the following eigenvalue equations:

* is an eigenvalue equation for position operator with eigenvalue in Dirac notation.
  + is an eigenvalue equation for position operator with eigenvalue in position representation.
  + is the position eigenfunction with eigenvalue in position representation and the position operator in position representation is (multiplication of the position space wavefunction by ).
* is the eigenvalue equation for momentum operator with eigenvalue in Dirac notation.
  + is the eigenvalue equation for momentum operator with eigenvalue in momentum representation.
  + is the momentum eigenfunction with eigenvalue in momentum representation and the momentum operator in momentum representation is (multiplication of the momentum space wavefunction by ).
  + is the eigenfunction of momentum with eigenvalue in position representation and the momentum operator in position representation is .
  + is the eigenvalue equation for momentum operator with eigenvalue in position representation.

1. You have not yet learned about the position operator in momentum representation. Which one of the following equations would help you identify the position operator in momentum representation?
2. , where is a generic state written in position representation.
3. , where is the position eigenfunction with eigenvalue in momentum representation.
4. , where is a generic state written in momentum representation.
5. , where is the momentum eigenfunction with eigenvalue in position representation.
6. The correct answer to the preceding question is (b). Consider the eigenvalue equation for the position operator in momentum representation, , in which is the position eigenfunction with eigenvalue in momentum representation. Which one of the following must be the position operator in momentum representation to satisfy the eigenvalue equation ?
7. Substitute in the answer you selected in the preceding question (for the position operator in momentum representation) into the eigenvalue equation for the position operator in momentum representation, , in which is the position eigenfunction with eigenvalue in momentum representation. Check whether the eigenvalue equation is satisfied. If not, correct your answer to the preceding question. Show **all** your work in the space provided below.

You now know the position operator and momentum operator in both the position and momentum representations. Fill in the table below with your results:

|  |  |  |
| --- | --- | --- |
|  | Position Representation | Momentum Representation |
| Position operator |  |  |
| Momentum operator |  |  |

1. Consider the following conversation between two students:

* Student 1: The position operator acting on a generic state written in Dirac notation without reference to a basis is . Suppose we choose a basis in which the eigenstates of position, , are chosen as basis vectors. is represented in position representation by taking the scalar product of with , like this: .
* Student 2: I don’t see how that is correct. How did the state vector turn into the position space wavefunction ?
* Student 1: We can insert the identity operator written in terms of position eigenstates into the expression , like this:

.

* Student 2: I see. So the position operator acting on a generic wavefunction in position representation just amounts to multiplication of by .

Do you agree with Student 1’s explanation and Student 2’s statement? Explain your reasoning.

1. Choose all of the following that correctly describe the momentum operator in position representation acting on a generic position space wavefunction i.e., ?
3. (I) only
4. (II) only
5. (I) and (IV) only
6. (I), (II), and (III) only
7. Consider the following conversation between Student A and Student B:

* Student A: The eigenvalue equation for momentum in position representation when the basis vectors are chosen to be the eigenstates of position, , is given by the expression . This is because , and since sandwiched in the middle is a number, we are left with .
* Student B: I agree with you. But we can also think about it as

, where is the momentum operator in position representation. Here, is a momentum eigenfunction with eigenvalue in position representation (which is a special type of position space wavefunction , where , a momentum eigenstate).

Do you agree with Student A and Student B? Explain why or why not.

Note: For a hermitian operator , the notation with between two vertical lines is the same as , i.e., since a hermitian operator can act forward or backwards on the state. If an operator is not hermitian (does not correspond to a physical observable), one should assume that the operator acts on the state after it (to the right of the operator) even if the notation is used.

1. Suppose a generic operator depends only on position and momentum operators. How is acting on a generic state , i.e., , represented in the position representation when basis vectors are chosen to be eigenstates of postion, ?
2. (I) and (IV) only
3. (II) and (III) only
4. (I), (II), and (III) only
5. All of the above.
6. Consider the following conversation between two students:

* Student 1: The correct answer for question 27 is (b). It is just like question 26, except question 26 is a special case of question 27.
* Student 2: How is similar to ?
* Student 1: Well, in question 26, is like because the operator corresponds to and state corresponds to state . Both and are operators acting on a state vector without reference to a basis. When basis vectors are chosen to be eigenstates of position, , and we take the scalar product of or each with , we obtain the respective operators written in position representation acting on the respective position space wavefunction. The operators and state vectors in each case are represented in position representation by and . Also, , which is a momentum eigenfunction with eigenvalue , is a special type of position space wavefunction .

Do you agree with Student 1’s explanation? Explain your reasoning.

1. Suppose a generic operator only depends on the position and momentum operators. How is acting on a generic state , i.e., , representated in the momentum representation when basis vectors are chosen to be eigenstates of momentum, ?
2. (I) and (II) only
3. (II) and (III) only
4. (II), (III), and (IV) only
5. All of the above
6. Consider the following conversation between Student A and Student B:

* Student A: Is the operator in the position representation, where is a function of position and momentum operators written in position representation, i.e., and , respectively?
* Student B: Yes. Suppose is the kinetic energy operator. In position representation, will be , if the particle is confined in one spatial dimension.
* Student A: And if is the kinetic energy operator in momentum representation, will be , since the momentum operator in the momentum representation. In other words, in the momentum representation, momentum operator acting on a state simply amounts to multiplication of by to yield , i.e., .

Do you agree with Student A, Student B, both, or neither? Explain your reasoning.

* **For a generic state , momentum space wavefunction and position space wavefunction**  **are a Fourier transform and inverse Fourier transform of each other, respectively.**

**Problem:** Show that the momentum space wave function is the Fourier transform of the position space wave function.

The problem has been broken down into several multiple choice questions to guide your solution.

1. Which one of the following is the correct expression for?
2. Using the completeness relation, insert a complete set of **position eigenstates** into the momentum space wave function, . What do you obtain?
3. The expression after you insert the complete set should be . What is ? You can go back to question **11** for help.
4. Rewrite the expression , using your answer to the preceding question in place of .
5. Rewrite your answer to the preceding questionreplacing with , since is the most common notation for the wave function in position representation.
6. The Fourier transform of a general function is *.* Compare this with your answer to the preceding question. The two expressions are analogous. *(Hint: Using the de Broglie relation, one can show that  , where p is the momentum and k is the wave number.)*

**Summary: Connecting Dirac notation with function space (position and momentum representation)**

* **Relationship between state vector and the wavefunction**
  + is the representation of the state vector in the position representation (when a complete set of eigenstates of position are chosen as basis vectors, i.e.,

) and is called the position space wavefunction.

* + is the representation of the state vector in the momentum representation (when a complete set of eigenstates of momentum are chosen as basis vectors, i.e.,

) and is called the momentum space wavefunction.

* + The position space wavefunction and momentum space wavefunction can be represented as infinite dimensional column vectors in the position and momentum representation, respectively.
  + The momentum space wavefunction and position space wavefunction are a Fourier transform and inverse Fourier transform of each other, respectively.
* **Position and momentum eigenstates**
  + The position eigenstates with eigenvalue can be written as:
    - in Dirac notation without reference to a basis.
    - in position representation.
    - in momentum representation.
  + The momentum eigenstates with eigenvalue can be written as:
    - in Dirac notation without reference to a basis.
    - in the position representation.
    - in the momentum representation.
* **in position and momentum representations**
  + For a generic state and generic operator which depends only on and :
    - is represented by in the **position** representation, which is by definition , where .
    - is represented by in the **momentum** representation, which is by definition , where .
  + Special cases of in position and momentum representations:
    - * Position representation:
      * Momentum representation:
      * Position representation:
      * Momentum representation:

|  |  |  |
| --- | --- | --- |
|  | Position representation | Momentum representation |
| Position eigenstates  (or position eigenfunctions) |  |  |
| Momentum eigenstates  (or momentum eigenfunctions) |  |  |

|  |  |  |
| --- | --- | --- |
|  | Position Representation | Momentum Representation |
| Position operator |  |  |
| Momentum operator |  |  |

|  |  |  |
| --- | --- | --- |
|  | Position Representation | Momentum Representation |
|  |  |  |
|  |  |  |
|  |  |  |

**Hamiltonian operator for a given quantum system and its eigenstates (energy eigenstates)**

1. For a given quantum system, what is  acting on an energy eigenstate (also represented by ) equal to?
2. , where *E* is the average of all possible energies of the system.
3. , where *En* is the energy of the *n*th state.
4. , , where *En* is the energy of the *n*th state.
5. None of the above. The Hamiltonian of a system must be known explicitly to determine the answer.
6. Consider the following conversation between Student A and Student B about the preceding question (the symbols have the same meaning as in the preceding question):

* Student A**:** The answer should be choice (d) for the previous question. The Hamiltonian describes the system, and we cannot determine what will happen without knowing explicitly.
* Student B**:** I agree. But this time we are given that are the energy eigenstates. Energy eigenstates are the eigenstates of the Hamiltonian . Any operator acting on one of its eigenstates will give the same state back with the corresponding eigenvalue.
* Student A: Right. So we do need to know the Hamiltonian of a system to calculate the energy eigenstates and eigenvalues .
* Student B: Yes. We would need the Hamiltonian for a given quantum system to calculate the energy eigenstates and eigenvalues explicitly, but we are asked to select an expression for the eigenvalue equation for . The expression in choice (c) is correct, even though we don’t know what the explicit ’s and *En*’s are without knowing the Hamiltonian .

With whom do you agree? Explain.

1. Consider the following conversation between Student A and Student B:

* Student A: The eigenvalue equation for a Hamiltonian without choosing a basis (e.g., position or momentum representation) is . can be written in position representation as . We can see this by taking the inner product of with , like this: . Using the fact that for any generic operator and state , can be written as .
* Student B: I disagree with you. We can’t write the eigenvalue equation for the Hamiltonian as because we don’t know the Hamiltonian explicitly. Also, we can’t write the Hamiltonian in terms of and as if we don’t know the Hamiltonian explicitly.

With whom do you agree? Explain your reasoning below.

**Problem:** Show that the expectation value of energy in a generic state of the system is

, where are the coefficients in the expansion . *En* is the eigenvalue for the *n*th energy eigenstate .

The problem has been broken down into several multiple choice questions to guide your solution.

1. Student A uses the completeness relation and inserts a complete set of energy eigenstates , into the expression for the expectation value , so that it becomes . Choose all of the following statements that are correct about the new expression .
   1. It is the same as the expression , because is equal to the identity operator.
   2. One can use and interchangeably based upon convenience because they are equal.
   3. is an operator, so it changes the expression when you insert it in the middle in . Thus, the new expression is not the same as the old expression.
2. (I) only
3. (II) only
4. (III) only
5. (I) and (II) only
6. Choose all of the following statements that are correct.
   1. , because and is a number that can be pulled out of the inner product.
   2. .
   3. (where ).
7. (I) and (II) only
8. (I) and (III) only
9. (II) and (III) only
10. All of the above.
11. Use your answers to the four preceding questions to show that .

**Summary of the Hamiltonian operator for a given quantum system, its eigenstates, and expectation value of energy:**

* The result of the Hamiltonian operator for a given system acting on an energy eigenstate state is given by the eigenvalue equation . In position representation, the eigenvalue equation for the Hamiltonian operator can be written as .

**Checkpoint**

Consider the following statement about a particle in a one dimensional infinite square well:

1. If the state of the system at time is , in which and are the lowest two energy eigenstates (ground state and first excited state with energy eigenvalues and , respectively), satisfies the eigenvalue equation .

Explain why you agree or disagree with this statement.

1. Work out for to check whether it satisfies the energy eigenvalue equation .

* The expectation value of energy in a generic state  is , where *Cn* = is the coefficient in the expansion and *En* is the eigenvalue of the *n*th energy eigenstate .

**Answer to Checkpoint**

a. Disagree. It is not an eigenstate of the Hamiltonian (not a stationary state).

b.

1. and or are not actually normalized because the inner products are and . and are not normalized because they diverge when or , respectively. It is actually the integral over all space of a Dirac delta function that equals 1, e.g.,

   . However, we will ignore these normalization issues for position and momentum eigenstates. [↑](#footnote-ref-1)