**Dirac Notation Basics**

* For all questions involving a generic operator corresponding to a physical observable , assume that it only depends on position and momentum .
* For a hermitian operator , the notation with between two vertical lines is the same as , i.e., since a hermitian operator can act forward or backwards on the state, i.e., . If an operator is not hermitian (does not correspond to a physical observable), one should assume that the operator acts on the state after it (to the right of the operator), even if notation is used.

**The goals of this tutorial are to help you learn that:**

* **The state of a quantum system is a vector**
  + that lies in an dimensional Hilbert space.
  + that has all possible information about the quantum system.
* **Scalar products (or inner products)** 
  + are defined as the component of one state along another state .
  + Are, in general, complex numbers (the number could have dimensions).
* **Hilbert Space**
  + A quantum state  is a vector in the Hilbert space.
  + The dimensionality of the Hilbert space is given by the number of linearly independent vectors that span the Hilbert space.
  + The eigenstates of a non-degenerate hermitian operator can be chosen as the basis vectors for the Hilbert space because they span the space.
* **Expansion of a state using a complete set of eigenstates**
  + A state can be written in terms of a linear superposition of a complete set of eigenstates {} of any hermitian operator .
  + The coefficients in the expansion, , are the components of the state along the direction of the eigenstates of a hermitian operator .
* **Probability of measuring an eigenvalue of a hermitian operator in a generic state** 
  + For orthonormal eigenstates {} with **discrete** eigenvalues ,  is the probability of measuring for an observable
  + For orthonormal eigenstates with **continuous** eigenvalues , is the probability of measuring the observable in a narrow range between and .
* **Expectation value of an operator in a generic state in terms of eigenstates and eigenvalues of** 
  + The expectation value of a hermitian operator with eigenstates {, } and **discrete** eigenvalues in a generic state is .
  + The expectation value of a hermitian operator with eigenstates and **continuous** eigenvalues in a generic state is
* **Completeness relation**
  + The completeness relation can be written as , where {} form an orthonormal basis for an *N* dimensional vector space. is the identity operator.
  + The completeness relation can be written as , where form an orthonormal basis for an infinite dimensional vector space.
  + The completeness relation is useful for decomposing a state vector into its components along each of the basis vectors.
* **Projection operator**
  + The projection operator acting on a state returns a vector in the direction of together with a number , which is the component of a state vector along the direction of the orthonormal basis vector .
  + The projection operator acting on a state returns a vector in the direction of together with a number , which is the component of a state vector along the direction of the orthonormal basis vector .

**State of the quantum mechanical system**

1. Choose all of the following statements that are correct.
   1. In Dirac notation, eigenstates of a physical observable are generally labeled by the corresponding eigenvalue. For example, position eigenstates are labeled by eigenvalues , and momentum eigenstates are labeled by eigenvalues .
   2. The quantum state written in Dirac notation,, lies in an abstract Hilbert space.
   3. The state contains all information one can obtain about the system at a given time.
2. (I) and (II) only
3. (II) and (III) only
4. (I) and (III) only
5. All of the above
6. Choose all of the following statements that are correct.
   1. The state vector in Dirac notation, , is an abstract vector without reference to a coordinate system.
   2. The infinite dimensional column vector when considered as a function of x is the wavefunction of the system at a given time. is obtained when the position eigenstates are chosen as the basis vectors to write state .
   3. The state vector and wavefunction have the same information, but  is a vector with position eigenstates as the coordinate axes and  for each *x* denotes the component of along the direction of .
      1. (I) and (II) only
      2. (I) and (III) only
      3. (II) and (III) only
      4. All of the above.

**Summary of state vectors:**

* The state of a quantum system is given by a vector in an abstract Hilbert space.
* The state contains all possible information about the quantum system at a given time.
* The state makes no reference to a coordinate system until the basis vectors are chosen.
* The infinite dimensional column vector when considered as a function of is the wavefunction of the system at a given time. is obtained when the position eigenstates are chosen as the basis vectors to write state .
* The State vector and wavefunction  have the same information, but  is a vector with position eigenstates as the coordinate axes and  for each *x* denotes the component of along the direction of .

**Scalar product (Inner product)**

1. The scalar product, or inner product, gives the component of a state, e.g., , along another state, e.g., . Choose all of the following notations that are correct for the scalar product that gives the component of state along state ?
   * 1. (I) only
     2. (II) only
     3. (III) only
     4. (I) and (II) only
2. Which one of the following equations is correct in general?
3. , where \* denotes complex conjugate.
4. , where \* denotes complex conjugate.
5. Since the wave function is normalizable, the scalar product of a normalized state vector with itself gives
   * 1. can be any finite number depending on the state.
     2. , where  is a phase factor that depends on the state.

**Summary of scalar products (inner products):**

* A scalar product of state with , , is defined as the component of state along state .
* A scalar product is not a vector. In general, the scalar product is a complex number.
* Interchanging the “bra” and “ket” states in a scalar product produces its complex conjugate: .
* The scalar product of a normalized state with itself gives 1, i.e., .

**Hilbert Space**

1. Which one of the following statements is true about the Hilbert space corresponding to a spin ½ system?
   * 1. The Hilbert space is two dimensional and the spin operator corresponding to each of the spin components has two eigenstates that form a complete set of basis vectors.
     2. The Hilbert space is three dimensional because the physical laboratory space is three dimensional and Hilbert space is a mathematical representation of the real world.
     3. The Hilbert space is infinite dimensional, because a finite dimensional space cannot be the Hilbert space for any quantum mechanical system.
     4. None of the above.
2. Choose all of the following statements that are correct about the eigenstates of an operator in a Hilbert space.
   1. An operator in a finite dimensional Hilbert space can have a finite number of discrete eigenvalues.
   2. An operator in an infinite dimensional Hilbert space can have infinitely many discrete eigenvalues.
   3. An operator in an infinite dimensional Hilbert space can have infinitely many continuous eigenvalues.
      1. (I) and (II) only
      2. (I) and (III) only
      3. (II) and (III) only
      4. All of the above.
3. Suppose is an observable for a given quantum system and its corresponding operator in the Hilbert space is . Choose all of the following statements that are correct.
4. must be a hermitian operator.
5. for all states and in the Hilbert space.

A. 1 only B. 2 only C. 3 only D. 1 and 2 only E. all of the above

1. “Any state vector in a Hilbert space can be expanded as a linear combination of a complete set of eigenstates of a hermitian operator.” Which one of the following does this statement imply? Note: Answer choices below may be correct statements but may not be implied by the statement in this question.
   * 1. All hermitian operators commute with each other and have simultaneous eigenstates.
     2. Eigenstates of a Hermitian operator can be chosen to be the coordinates (basis vectors) in the Hilbert space.
     3. All hermitian operators have real eigenvalues that correspond to results of measurements in physical space.
     4. The given statement is incorrect. The correct statement should read “Any vector in Hilbert space can only be expanded as a linear superposition of a complete set of energy eigenstates (eigenstates of the Hamiltonian operator).”
2. “Any state vector in the Hilbert space can be expanded as a linear superposition of a complete set of eigenstates of a hermitian operator” is a correct statement. Choose all of the following that can be examples of the mathematical representation of this statement.
   1. , where are energy eigenstates for a given quantum system and are appropriate expansion coefficients.
   2. , where are position eigenstates and are appropriate expansion coefficients.
   3. , where are energy eigenstates and are appropriate expansion coefficients.
      1. (I) only
      2. (I) and (II) only
      3. (II) and (III) only
      4. All of the above.

**Summary of Hilbert Space:**

* Quantum state vectors are vectors in the Hilbert space.
* State vectors can be expanded as a linear superposition of a complete set of eigenstates of a hermitian operator.
* The dimensionality of the Hilbert space is given by the number of linearly independent vectors in the space. The eigenstates of a hermitian operator span the space which means that they form a complete set of basis vectors for the Hilbert space.
* When the measurement of an observable is performed in physical space, the values one measures are the eigenvalues of the corresponding hermitian operator.

**Checkpoint 1**

Consider the following conversation between student A and student B:

* Student A: The Hilbert space for a particle interacting with a one dimensional infinite square well is infinite dimensional. Also, the position eigenstates form a complete set of basis vectors for the space and the position of the particle has infinitely many values within the width of the square well.
* Student B: I disagree. The Hilbert space for a particle interacting with a one dimensional infinite square well is one dimensional, because the well is one dimensional and the particle is confined in one dimension.

Which student, if either, do you agree with and why?

**Expansion of a state vector in terms of a complete set of eigenstates**

1. Earlier you learned that any vector in the Hilbert space can be expanded as a linear superposition of a complete set of eigenstates of a hermitian operator . For an operator with eigenstates {} (which form an orthonormal basis for an dimensional vector space) and **discrete** eigenvalues , choose all of the following statements that are correct about the coefficients in the expansion

.

* 1. To find , we take the scalar product with an eigenstate . Then,

.

* 1. To find , we take the scalar product with an eigenstate . Then,

.

* 1. The coefficient for a particular eigenstate in the expansion is related to the probability of measuring the corresponding eigenvalue when a measurement of observable is made in the state .
     1. (I) and (II) only
     2. (I) and (III) only
     3. (II) and (III) only
     4. All of the above.

1. Consider the following conversation between Student A and Student B.

* Student A: In the preceding question, statement (II) seems to make more sense than statement (I), because we are using the expansion as opposed to .
* Student B: But *m* and *n* are just “dummy” indices. They both can range from 1 to *N*, where *N* is the dimension of the Hilbert space.
* Student A: What is the point of changing the label from to in the expansion of in statement (I)?
* Student B: If you take the scalar product of with an eigenstate with the same index , like in statement (II), you obtain . You end up with a sum over all ’s. This is incorrect. You must take the inner product with a different “dummy” indexed state , so that you get . Instead of , you obtain which gets rid of the summation. This is the correct answer, which is just a single coefficient , not a sum .
* Student A: I see. If we choose a different dummy index for state when taking the inner product, we get , which gets rid of the sum over .

Do you agree with Student B’s explanation? Explain why or why not.

1. Consider the following conversation between Student A and Student B.

* Student A: For an operator with a discrete eigenvalue spectrum, such as energy, we can talk about measuring each of the eigenvalues. We can calculate the probabilities for measuring each of them individually.
* Student B: I agree. But for an operator that has a continuous eigenvalue spectrum, like position or momentum, we should talk about measuring a value in a narrow range. For example, the probability of measuring position between  is .

Do you agree with Student A and/or Student B? Explain your reasoning.

1. For an arbitrary physical observable with **discrete** eigenvalues and eigenstates , where

, write the **probability** of measuring eigenvalue as a result of a measurement of performed when the system is in the state .

1. Earlier you learned that any vector in the Hilbert space can be expanded as a linear superposition of a complete set of eigenstates of a hermitian operator . In the case of an operator with **continuous** eigenvalues and eigenstates (which form an orthonormal basis for an infinite dimensional vector space), choose all of the following statements that are correct about the coefficients in the expansion

? *(Hint: This is similar to question 11, except the eigenvalues are continuous).*

* 1. To find ,we take the scalar product with an eigenstate . Then,

.

* 1. Physically, the coefficient is the component of the state along the direction of the eigenvector (with eigenvalue .
  2. The coefficient in the expansion is related to the probability of measuring the eigenvalue when observable is measured in the state .
     1. (I) and (II) only
     2. (I) and (III) only.
     3. (II) and (III) only
     4. All of the above.

1. Earlier, you learned that for an operator corresponding to a physical observable with eigenstates {, } with **discrete** eigenvalues
2. is the **probability amplitude** for measuring if we measure observable
3. is the **probability** for measuring if we measure observable .

Before Dirac notation was introduced, physicist Max Born interpreted the probabilistic nature of quantum mechanics and proposed the following statements for the continuous case for observable which has a continuous eigenvalue spectrum:

1. is the **probability density amplitude** for measuring position.
2. is the **probability density** for measuring position.
3. is the **probability** of finding the particle in the narrow range between  when position of the particle is measured.

Write each of these expressions (probability density amplitude, probability density, and probability of measuring position in a narrow range between and ) in Dirac notation.

1. Dirac extended Born’s interpretation to apply to measurements of not only position but **any** physical observable. Keeping in mind your answers to the two preceding questions, for an arbitrary physical observable with eigenstates with **continuous** eigenvalues, write the **probability** of measuring observable between and as a result of a measurement performed when the system is in the state .
2. The **expectation value** of an operator is the average value of the observable measured over many identical experiments performed on identically prepared systems in state . For a general quantum mechanical hermitian operator , the expectation value is represented by . If has **discrete** eigenvalues and eigenstates where , let’s write in terms of the eigenstates and eigenvalues .
3. Write as a linear superposition of the eigenstates of .
4. Consider the following conversation between two students:

* Student A: If we write as a linear superposition of the eigenstates of , we obtain

, where is the expansion coefficient.

* Student B: I agree with you. But we know that the expansion coefficients, , are the eigenvalues of the operator . So we can write as a linear superposition of the eigenstates of like this: .

With whom do you agree? Explain your reasoning.

1. The linear superposition of in terms of the eigenstates of can be written as

, where is the expansion coefficient and gives the component of the state along the direction of the nth eigenstate Write explicitly in terms of and .

1. What is the probability of measuring when you measure observable in the state ?
2. Consider the following conversation between Student A and Student B:

* Student A: is the probability of measuring when you measure observable in the state . The expectation value is the average value of a large number of measurements performed on identically prepared systems. Since we know the probability of measuring each eigenvalue of the operator , the expectation value is .
* Student B: No. You cannot think about expectation value physically as an average of a large number of measurements on identically prepared systems. We must use our expansion , to calculate the expectation value .

With whom, if either, do you agree? Explain your reasoning.

1. Student A is correct. The expectation value is the average value of a large number of measurements on identically prepared systems, which can be represented mathematically by the equation . But let’s follow Student B’s method using the expansion to prove that the equation , suggested by Student A, is correct. Act with on the state . What do you obtain?
2. So far, we have . Using your answer from part (c), insert what you obtained for into .
3. We now have . Now take the inner product of with a “bra” state to find the **expectation value** . Does your answer agree with Student A’s statement from part (e)? If not, go back and check your work with a partner to obtain the equation for the expectation value of observable in terms of its complete set of eigenstates { and eigenvalues , i.e., .
4. Repeat the calculation for the **expectation value** of an operator with eigenstates (which form a basis in an infinite dimensional vector space) with **continuous** eigenvalues .
5. Suppose an operator corresponding to a physical observable has eigenstates {, } with discrete eigenvalues . Which of the following are correct about the expression ?
6. is equal to a real number.
7. is equal to a complex number that does have an imaginary part.
8. (II) only
9. (III) only
10. (I) and (II) only
11. None of the above.
12. Consider the following conversation between two students about the preceding question:

* Student A: I don’t see how .
* Student B: Let me show you.

1. We can start with , since any state vector can be normalized to 1.
2. Insert the identity operator, written in terms of the eigenstates {, } of the operator , like this: .
3. Using the fact that , we can write

.

* Student A: I see. Is there any physical significance to ?
* Student B: Yes. is the probability amplitude and is the probability for measuring if we measure observable .
* Student A: So based on the mathematical expression , the probabilities of measuring different eigenvalues when we measure the observable must add up to 1.

Do you agree with Student A and Student B? Explain your reasoning.

**Summary of the expansion of a state vector in terms of a complete set of eigenstates:**

* We can write the state vector in terms of a linear superposition of the energy eigenstates, position eigenstates, or eigenstates of any other hermitian operator since each of them spans the space.
  + If has eigenstates ( ) with **discrete** eigenvalues , then

, in which is the component of the state along , i.e., .

* + If has eigenstates with **continuous** eigenvalues , then , in which is the component of the state along the eigenstate , i.e., .
  + If a hermitian operator has a discrete eigenvalue spectrum, the expansion of a state vector in terms of the eigenstates of the hermitian operator is a sum. If a hermitian operator has a continuous eigenvalue spectrum, the expansion of a state vector vector in terms of the eigenstates of the hermitian operator is an integral.
  + For **discrete** eigenvalues ,  is the **probability** of measuring for an observable when the system is in the state .
  + For **continuous** eigenvalues , is the **probability** of measuring an observable in a narrow range between and when the system is in the state .
* The **expectation value** of a hermitian operator in a generic state is the average value of the observable measured over many identical experiments performed on identically prepared systems in state .
  + In a generic state , the expectation value for an operator with eigenstates () with **discrete** eigenvalues is
  + In a generic state , the expectation value for an operator with eigenstates with **continuous** eigenvalues is

**Completeness Relation**

The completeness relation can be written in terms of orthonormal eigenstates of an operator with discrete eigenvalues . Mathematically, the completeness relation is , where {} is an orthonormal basis for an *N* dimensional Hilbert space and  is the identity operator which can be represented by an identity matrix. Completeness (of a set of basis vectors, e.g., eigenstates of an operator corresponding to a physical observable) means that an arbitrary state vector can be written in terms of the complete set of basis vectors.

1. a) Act with the identity operator , written in terms of orthonormal eigenstates of an operator with discrete eigenvalues , on an arbitrary state vector . What do you obtain?

b) Explain your results from part (a) in a sentence.

1. So far we have If , which one of the following is the correct expression for the coefficients along the state in the expansion of ?

The identity operator acting on an arbitrary state is . This shows that a generic state can be written in terms of a complete set of eigenstates which span the Hilbert space. We often use the completeness relation to decompose a generic state into its components along each of the basis vectors (eigenstates of a hermitian operator can be chosen to be the basis vectors in the Hilbert space).

1. Re-calculate expectation value of in state , , by using the completeness relation inserted into the expression and compare to your answer for question 18 part (h).
2. The completeness relation can also be written in terms of the eigenstates of an operator with a **continuous** eigenvalue spectrum. The completeness relation corresponding to a hermitian operator with eigenstates with continuous eigenvalues is , where is an infinite dimensional identity matrix.
   * + 1. What is the result of (completeness relation written in terms of a complete set of eigenstates ) acting on ?
       2. So far have . Explain whether is a number, operator, or vector.
       3. Consider the following conversation between Student A and Student B:

* Student A: The component of along the basis vector is , which is a number. So we are free to move in the integral . So .
* Student B: I disagree with you. We cannot simply move around inside the integral, like this .

With whom, if either, do you agree? Explain your reasoning.

* + - 1. Using your answers to the preceding parts (a-c), what is the expression for an arbitrary state written in terms of continuous eigenstates of a hermitian operator and numbers ?
      2. Use your answers to the preceding parts (a-d) to calculate the expectation value in terms of eigenstates with continuous eigenvalues of a hermitian operator .

1. Which one of the following relations is correct about an operator with eigenstates {} (which form an orthonormal basis for an dimensional vector space) and **discrete** eigenvalues ?
2. To check your answer to the preceding question, you must show that the operator acting on any generic state gives the same result as the right hand side of the expression in the preceding question.
3. Act with the operator on a generic state , like this: . Now insert the identity operator, written in terms of the orthonormal eigenstates {} of the operator , between the operator and generic state .
4. So far, you should have . We can think of like this:

, such that the terms in the curly brackets must be equal. So the operator .

1. Using your answers to the preceding parts (a)-(c), determine the expression for a hermitian operator with eigenstates with **continuous** eigenvalues in terms of the eigenstates and eigenvalues .

**Summary of the completeness relation:**

* A complete set of orthonormal eigenstates of a hermitian operator with discrete or continuous eigenvalues can be used to write the completeness relation.
* The completeness relation is useful for decomposing a state vector into its components along each of the basis vectors.
  + The completeness relation for basis vectors with a **discrete** eigenvalue spectrum is , where {} is an orthonormal basis for an *N* dimensional vector space (e.g., formed with a complete set of eigenstates with eigenvalues of an operator corresponding to an observable ).
  + The completeness relation for basis vectors with a **continuous** eigenvalue spectrum is

, where is an orthonormal basis for an infinite dimensional vector space (e.g., formed with a complete set of eigenstates with eigenvalues of an operator corresponding to an observable ).

* The identity operator doesn’t change the vector it acts on.
* An operator with a complete set of orthonormal eigenstates {} with **discrete** eigenvalues can be written as .
* An operator with a complete set of orthonormal eigenstates with **continuous** eigenvalues can be written as .

**Checkpoint 2**

Choose all of the following statements which are correct for a given quantum system about an operator that corresponds to a physical observable with discrete eigenvalues and eigenstates :

1. , where are expansion coefficients.
2. is the probability of measuring in a generic state .
3. is the probability of measuring in a generic state .
4. (I) only
5. (II) only
6. (I) and (II) only
7. (I) and (III) only
8. None of the above.

**Projection Operator**

1. Which one of the following statements is correct about the expression, where } form an orthonormal basis for an *N* dimensional vector space?
   * 1. is equal to the number 1.
     2. is a scalar, but one cannot determine what number it is equal to without knowing what is explicitly.
     3. is an outer product, so it is an operator.
     4. is a vector.
2. Act on a generic state with the operator . That is, . Which one of the following statements correctly describes what you obtain?
   * 1. You get back the same state , because is the identity operator.
     2. You get the projection of along the direction of . is the component of along the direction of . The vector , which multiplies the coefficient , gives the direction of the projected vector.
     3. You get the same state back, with the corresponding eigenvalue.
     4. It cannot be determined from the given information. The state has to be given explicitly in position representation for a given quantum system to be able to calculate the answer.
3. Consider the following conversation between Student A and Student B.

* Student A: I thought that was equal to the identity operator. Wasn’t that what we had learned earlier in this tutorial? How is it that the same expression is the identity operator and the projection operator at the same time?
* Student B: The expression that was equal to the identity operator was , where there is a sum over a complete set of basis vectors. Applying that on a state would give the same state back. An example of a projection operator is . Acting with on a state gives the projection of that state along the direction of as follows:

The vector multiplying

the inner product

Inner product of with equals the component of along the direction of

Do you agree with Student B’s explanation? Explain why or why not.

**Summary of the projection operator:**

* The projection operator returns the component of a state vector along the direction of a vector .
* Unlike the identity operator, the projection operator acting on a state vector need not return the same state vector back.
* The projection operator formed with orthonormal eigenstates of a hermitian operator with discrete or continuous eigenvalues has a similar affect on a generic state as follows:
  + is a projection operator, e.g., written in terms of orthonormal eigenstates with **discrete** eigenvalues of an operator . The projection operator projects a generic state along the direction of vector .
  + is a projection operator, e.g., written in terms of orthonormal eigenstates with **continuous** eigenvalues of an operator . The projection operator projects a generic state along the direction of vector .

**Checkpoint 3**

Assume we have a generic vector in a three dimensional Hilbert space where are complex numbers and form an orthonormal basis.

1. Write the projection operator that projects along the basis vector .
2. Use the projection operator you constructed to find the components of state along the direction .
3. Act with the identity operator, written in terms of the basis vectors on the vector .
4. Write down, in your own words, the difference between a projection operator and the identity operator based upon your answers to the preceding parts.

**Answers to Checkpoints**

Checkpoint 1:

Student A. There are an infinite number of values of position in a one dimensional infinite square well.

Checkpoint 2:

(d)

Checkpoint 3:

a.

b.

c.

d. The identity operator acting on a state gives the same state back. A projection operator acting on a state gives a component of the state along a basis vector times a basis vector.