**Dirac Notation Warm-Up: Getting Acquainted with Dirac Notation in a Familiar Context**

The goals of the Dirac Notation Warm-Up are to help you use a familiar context to:

* Learn how to write a vector in a given vector space using Dirac notation.
* Learn how to write the scalar product (or inner product) using bra and ket vectors and recognize that the scalar product is a number (it can have dimensions).
* Learn how to write the components of a vector along a complete set of orthonormal basis vectors using scalar products written in Dirac notation.
* Understand the difference between bra and ket vectors
	+ Ket vectors can be written as column matrices in a given basis.
	+ Bra vectors can be written as row matrices in a given basis.
* Understand the relationship between vector components in different orthonormal bases if the basis vectors are changed from one orthonormal basis to another.
* Learn how to write the outer product using bra and ket vectors and recognize that:
	+ An outer product is a linear operator.
	+ An outer product is equivalent to an $N×N$ square matrix in any given basis, where $N$ is the dimension of the vector space.
* Compute the identity operator in an $N$ dimensional vector space and recognize that:
	+ The identity operator is a square matrix with 1’s along the diagonal, regardless of which basis is chosen.
	+ The identity operator acting on a vector does not change the vector.
	+ The identity operator is found by taking the outer product of each normalized basis vector with itself then summing over them.
	+ Completeness relation: for a given basis, summation over all the outer products of each orthonormal basis vector with itself is the identity operator.
	+ The completeness relation can be used to decompose a vector into its components along the chosen orthonormal basis vectors.
* Understand that the projection operator formed from the outer product of a basis vector with itself acting on a generic vector returns the basis vector multiplied by the component of the generic vector along that basis vector.

**Dirac notation** is used extensively in quantum mechanics in connection with state vector $\left.\left|Ψ(t)\right.\right〉$ which:

* Lies in an abstract vector space (Hilbert space)
* Has all possible information about the state of the quantum system at a given time $t$.

Here we familiarize you with Dirac notation in the context of a familiar vector, force $\vec{F}$, in a real physical three dimensional vector space. Just as force $\vec{F}$ can be represented as a vector (with a magnitude and direction) in the three dimensional real physical space we live in, the state vector $\left.\left|Ψ\right.\right〉$ can be represented as a vector in an abstract vector space.

In introductory physics, you learned that it is often convenient to choose an orthonormal coordinate system or “basis” and break up (or decompose) force into components along the directions of “basis vectors.” In three dimensions, we typically name the three mutually orthogonal directions$ x$, $y$, and$ z$ and the unit vectors (or normalized basis vectors) along those directions $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively.

Then, we can write the force $\vec{F}$ as

$\vec{F}=a\hat{i}+b\hat{j}+c\hat{k}$ (1)

where $a, b, and c$ are the components of the force $\vec{F}$ along $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively.

In the familiar scalar (dot or inner) product notation:

$a=\hat{i}∙\vec{F}$ $b=\hat{j}∙\vec{F}$ $c=\hat{k}∙\vec{F}$ (2)

Let’s introduce the **Dirac notation** in which “ket” vectors are written as $\left|\left.F\right〉\right.$ and $\left|\left.i\right〉\right.$ rather than $\vec{F}$ and $\hat{i}$, respectively, and the scalar product is written as $\left⟨F\right⟩$ rather than $\hat{i}∙\vec{F}$. $\left〈\left.F\right|\right.$ and $\left〈\left.i\right|\right.$ are called “bra” vectors, which lie in the “dual” space. The scalar product $\left⟨F\right⟩$, which gives the component of a “ket” vector $\left|\left.F\right〉\right.$ along the direction of the “bra” vector $\left〈\left.i\right|\right.$, is often called a “bracket.” Hence, in Dirac notation

**Checkpoint 1**:

Consider the following statements from Student A and Student B.

* Student A: $\left(a\hat{i}+b\hat{j}\right)∙\left(c\hat{i}+d\hat{j}\right)=ac+bd$, which is a number, or scalar.
* Student B: I disagree. $\left(a\hat{i}+b\hat{j}\right)∙\left(c\hat{i}+d\hat{j}\right)=ac\hat{i}+bd\hat{j}$, which is a vector.

Explain why agree or disagree with each student.

$\left|\left.F\right〉\right.=a\left|\left.i\right〉\right.+b\left.\left|j\right.\right〉+c\left.\left|k\right.\right〉$ (3)

$a=\left⟨F\right⟩$ $b=\left⟨F\right⟩$ $c=\left⟨F\right⟩$ (4)



Figure 1

**Checkpoint 2:**

1. Consider the following statements from Student A and Student B.
* Student A: $\left⟨F\right⟩$ is a scalar which gives the magnitude of the force along the $x$ direction.
* Student B: I do not agree. $\left⟨F\right⟩$ is a vector which points along the $x$ direction.

Explain why you agree or disagree with each statement.

1. In Figure 1 (shown above), a force vector $\left|\left.F\right〉\right.$ with a magnitude of 5 N acts on an object in the plane of the paper ($z$-axis is perpendicular to the plane of the paper). The coordinate system (and therefore basis (unit) vectors along the the $x$, $y$, and $z$ axes) is chosen such that the force $\left|\left.F\right〉\right.$ makes a 30° angle with the $x$ axis. Find $\left⟨F\right⟩$, $\left⟨F\right⟩$, and $\left⟨F\right⟩$.

Note that the choice of coordinate system or “basis” is up to us and it may be convenient to choose another orthononormal basis $x^{'}$, $y^{'}$, $z^{'} $with unit vectors$ \hat{i'}$, $\hat{j'}$, and $\hat{k'}$, respectively (e.g., in inclined plane problems, it may be convenient to choose basis vectors to be parallel and perpendicular to the incline rather than vertical and horizontal). In the new basis, force $\vec{F}$ can be written in the traditional notation and Dirac notation respectively as

$\vec{F}=a'\hat{i}'+b'\hat{j}'+c'\hat{k}'$ (5)

$\left|\left.F\right〉\right.=a'\left|\left.i^{'}\right〉\right.+b'\left.\left|j^{'}\right.\right〉+c'\left.\left|k'\right.\right〉$ (6)



Figure 2

**Checkpoint 3:**

1. In Figure 2 (shown above), a force vector $\left|\left.F\right〉\right.$ with a magnitude of 5 N acts on an object in the plane of the paper. The coordinate system (and therefore basis vectors along the $x$, $y$, and $z$ axes) are chosen such that the force $\left|\left.F\right〉\right.$ makes a 90° angle with the $x$ axis (the identical coinciding $z$ and $z’$ axes are perpendicular to the plane of the paper). Find $\left⟨F\right⟩$, $\left⟨F\right⟩$, and $\left⟨F\right⟩$.
2. In Figure 2, a force vector $\left|\left.F\right〉\right.$ with a magnitude of 5 N acts on an object in the plane of the paper. Basis vectors (unit vectors along the $x’$, $y’$, and $z’$ axes) are chosen such that the force $\left|\left.F\right〉\right.$ makes a 45° angle with the $x’$ axis (the identical coinciding $z$ and $z’$ axes are perpendicular to the plane of the paper). Find $\left⟨F\right⟩$, $\left⟨F\right⟩$, and $\left⟨F\right⟩$.

**Using matrix form once a basis (coordinate system) has been chosen**

After we have chosen a set of basis vectors, it may be convenient to write the “ket” force vector in Eq. (3) as a column matrix $\left|\left.F\right〉\right.≐\left(\begin{matrix}\left⟨F\right⟩\\\left⟨F\right⟩\\\left⟨F\right⟩\end{matrix}\right)=\left(\begin{matrix}a\\b\\c\end{matrix}\right)$ (The $≐$ sign is defined on next page). For the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation chosen, the normalized basis vectors (unit vectors along directions of orthonormal coordinate axes) are

 $\left.\left|i\right.\right〉≐\left(\begin{matrix}1\\0\\0\end{matrix}\right)$ $\left.\left|j\right.\right〉≐\left(\begin{matrix}0\\1\\0\end{matrix}\right)$ $\left.\left|k\right.\right〉≐\left(\begin{matrix}0\\0\\1\end{matrix}\right)$ (7)

Note that in $\left|\left.F\right〉\right.≐\left(\begin{matrix}a\\b\\c\end{matrix}\right)$, $a, b, and c$ are components of vector $\left|\left.F\right〉\right.$ along the directions of basis vectors (unit vectors along the coordinate axes) and instead of an equality sign we choose the $≐$ sign to note that the equality is valid only with respect to a chosen basis. The “bra” vectors (used in the scalar product to find the component of $\left|\left.F\right〉\right.$ along the direction of the basis vectors, e.g., Eq. (4)) can be written as row matrices

$\left〈\left.i\right|≐(\begin{matrix}1&0&0\end{matrix})\right.$ $\left〈\left.j\right|≐\left(\begin{matrix}0&1&0\end{matrix}\right)\right.$ $ \left〈\left.k\right|≐(\begin{matrix}0&0&1\end{matrix})\right.$

(Using traditional notation, column matrix form can be introduced as follows $\vec{F}≐\left(\begin{matrix}\hat{i}∙\vec{F} \\j∙\vec{F} \\\hat{k}∙\vec{F} \end{matrix}\right)=\left(\begin{matrix}a\\b\\c\end{matrix}\right)$.)

We can verify that scalar products work out as expected by multiply matrices:

$$a=\left⟨F\right⟩=\left(\begin{matrix}1&0&0\end{matrix}\right)\left(\begin{matrix}a\\b\\c\end{matrix}\right)$$

$$b=\left⟨F\right⟩=(\begin{matrix}0&1&0\end{matrix})\left(\begin{matrix}a\\b\\c\end{matrix}\right)$$

$$c=\left⟨F\right⟩=(\begin{matrix}0&0&1\end{matrix})\left(\begin{matrix}a\\b\\c\end{matrix}\right)$$

$$1=\left⟨i\right⟩=\left(\begin{matrix}1&0&0\end{matrix}\right)\left(\begin{matrix}1\\0\\0\end{matrix}\right)$$

$0=\left⟨j\right⟩=(\begin{matrix}1&0&0\end{matrix})\left(\begin{matrix}0\\1\\0\end{matrix}\right)$, etc.

We have shown that $a=\left⟨F\right⟩$. But what is $\left⟨i\right⟩$? Generally, $\left〈\left.F\right|=\left(\begin{matrix}a^{\*}&b^{\*}&c^{\*}\end{matrix}\right)\right.$, where the asterisk denotes the complex conjugate of $a, b, and c$. So $\left⟨i\right⟩=\left(\begin{matrix}a^{\*}&b^{\*}&c^{\*}\end{matrix}\right)\left(\begin{matrix}1\\0\\0\end{matrix}\right)=a^{\*}$. But in mechanics, the components $a, b, and c$ of the force vector $\left|\left.F\right〉\right.$ are real numbers, so $a^{\*}=a$. In quantum mechanics, however, the components of state vector $\left|\left.Ψ\right〉\right. $ along the orthonormal basis vectors are in general complex numbers.

**Checkpoint 4:**

1. Consider the following statements from Student A and Student B.
* Student A: In $\left|\left.F\right〉\right.≐\left(\begin{matrix}a\\b\\c\end{matrix}\right)$, $a$, $b$, and $c$ have fixed values regardless of what basis we have chosen.
* Student B: That is not true. You can write the force vector as a column matrix only after you’ve chosen a basis or coordinate system. $a, b, and c$ give the components of force $F$ along the directions of basis vectors $\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, and $\left.\left|k\right.\right⟩$, respectively. If you choose a different basis, these components will be different.

Explain why you agree or disagree with Student A and Student B.

1. Do $\left⟨F\right⟩$ and $\left⟨j\right⟩$ differ from each other if $\left⟨F\right⟩$ is a real number? Will they be different if $\left⟨F\right⟩$ is a complex number?



Figure 3

As noted previously, we can choose another orthonormal basis (coordinate axes) if that is convenient for solving a problem. For example, the normalized basis vectors $\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, and $\left.\left|k'\right.\right⟩$ for the three dimensional vector space (unit vectors along directions of new orthogonal coordinate axes) are shown in Figure 3. (Note: The $z$ and $z^{'}$ axes are perpendicular to the plane of the paper.)

**In the {**$\left.\left|i\right.\right⟩$**,** $\left.\left|j\right.\right⟩$**,** $\left.\left|k\right.\right⟩$**} representation**, $\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$ can be represented as follows:

$\left.\left|i'\right.\right〉≐\left(\begin{matrix}\left⟨i^{'}\right⟩\\\left⟨i^{'}\right⟩\\\left⟨i^{'}\right⟩\end{matrix}\right)=\left(\begin{matrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{matrix}\right) \left.\left|j^{'}\right.\right〉≐\left(\begin{matrix}\left⟨j^{'}\right⟩\\\left⟨j^{'}\right⟩\\\left⟨j^{'}\right⟩\end{matrix}\right)=\left(\begin{matrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{matrix}\right)$ $\left.\left|k'\right.\right〉≐\left(\begin{matrix}\left⟨k^{'}\right⟩\\\left⟨k^{'}\right⟩\\\left⟨k^{'}\right⟩\end{matrix}\right)=\left(\begin{matrix}0\\0\\1\end{matrix}\right)$. (8)

These three equations can be combined into one equation involving a $3×3$ rotation matrix that relates the basis vectors in the {$\left.\left|i\right.'\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$} representation to the basis vectors in the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation:

$$\left(\begin{matrix}\left.\left|i'\right.\right⟩\\\left.\left|j'\right.\right⟩\\\left.\left|k'\right.\right⟩\end{matrix}\right)=\left(\begin{matrix}\left⟨i'\right⟩&\left⟨i'\right⟩&\left⟨i'\right⟩\\\left⟨j'\right⟩&\left⟨j'\right⟩&\left⟨j'\right⟩\\\left⟨k'\right⟩&\left⟨k'\right⟩&\left⟨k'\right⟩\end{matrix}\right)\left(\begin{matrix}\left.\left|i\right.\right⟩\\\left.\left|j\right.\right⟩\\\left.\left|k\right.\right⟩\end{matrix}\right)$$

On the other hand, **in the {**$\left.\left|i\right.'\right⟩$**,** $\left.\left|j'\right.\right⟩$**,** $\left.\left|k'\right.\right⟩$**} representation**, $\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$ can be represented as follows:

$\left.\left|i'\right.\right〉≐\left(\begin{matrix}\left⟨i^{'}\right⟩\\\left⟨i^{'}\right⟩\\\left⟨i^{'}\right⟩\end{matrix}\right)=\left(\begin{matrix}1\\0\\0\end{matrix}\right)$ $\left.\left|j'\right.\right〉≐\left(\begin{matrix}\left⟨j^{'}\right⟩\\\left⟨j^{'}\right⟩\\\left⟨j^{'}\right⟩\end{matrix}\right)=\left(\begin{matrix}0\\1\\0\end{matrix}\right)$ $\left.\left|k'\right.\right〉≐\left(\begin{matrix}\left⟨k^{'}\right⟩\\\left⟨k^{'}\right⟩\\\left⟨k^{'}\right⟩\end{matrix}\right)=\left(\begin{matrix}0\\0\\1\end{matrix}\right)$. (9)

x

x’

y

y’

30°

30°

**Checkpoint 5:**

1. In the figure shown above, what are the basis vectors$\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$ in the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation?
2. In the figure shown above, write the rotation matrix which relates the basis vectors in the

{$\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$} representation to the basis vectors in the in the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation.

1. In the figure above, what are the basis vectors $\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$ in the {$\left.\left|i\right.'\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$} representation?
2. Write a rotation matrix which relates the basis vectors in the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation to the basis vectors in the {$\left.\left|i\right.'\right⟩$, $\left.\left|j'\right.\right⟩$, $\left.\left|k'\right.\right⟩$} representation in which the matrix elements are written in Dirac notation. Calculate the numerical values of each matrix element in the $3×3$ rotation matrix for the case shown in the figure above.
3. If the force vector $\left|\left.F\right〉\right.$ written in the {$\left.\left|i\right.'\right⟩$, $\left.\left|j\right.'\right⟩$, $\left.\left|k'\right.\right⟩$} representation is $\left|\left.F\right〉\right.≐\left(\begin{matrix}a'\\b'\\c'\end{matrix}\right)$, what are $a^{'}$, $b^{'}$, and $c^{'}$ in Dirac notation?

**Outer product of vectors and completeness relation**

Linear operators act on vectors to give back another vector whose magnitude and direction may be different. The outer product of a “ket” vector $\left|\left.F\right〉\right.$ with a “bra” vector $\left〈\left.G\right|\right.$ is written as $\left|\left.F\right〉\right.\left〈\left.G\right|\right. $and is an example of a linear operator. For example, the operator $\left|\left.F\right〉\right.\left〈\left.G\right|\right.$ can act on the vector $\left|\left.P\right〉\right.$ and result in a vector $\left|\left.F\right〉\right.$ along with a complex number $C=\left⟨P\right⟩$ which is the component of the vector $\left|\left.P\right〉\right.$ along the $\left|\left.G\right〉\right.$ direction:

 $(\left|\left.F\right〉\right.\left⟨)\left|P\right.\right⟩=\left|\left.F\right〉\right.\left(\left⟨P\right⟩\right)=C\left|\left.F\right〉\right.$. (10)

We learned that after choosing a basis, we can write “bra” and “ket” vectors as row and column matrices. In the chosen basis (coordinates), the outer products become $N×N$ square matrices where N is the dimensionality of the vector space. For example, in a 3 dimensional vector space, in a particular basis, if $\left|\left.F\right〉\right.≐\left(\begin{matrix}a\\b\\c\end{matrix}\right)$ and $\left|\left.G\right〉\right.≐\left(\begin{matrix}d\\e\\f\end{matrix}\right)$, then

$\left|\left.F\right〉\right.\left〈\left.G\right|\right.≐\left(\begin{matrix}a\\b\\c\end{matrix}\right)\left(\begin{matrix}d^{\*}&e^{\*}&f^{\*}\end{matrix}\right)=\left(\begin{matrix}ad^{\*}&ae^{\*}&af^{\*}\\bd^{\*}&be^{\*}&bf^{\*}\\cd^{\*}&ce^{\*}&cf^{\*}\end{matrix}\right)$,

which is a $3×3$ matrix. Note that each matrix element of the outer product $\left|\left.F\right〉\right.\left〈\left.G\right|\right.$ (such as $ad^{\*}, ae^{\*}$, etc.) is a number.

**Checkpoint 6:**

1. In the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation, the normalized basis vectors are chosen as $\left.\left|i\right.\right〉≐\left(\begin{matrix}1\\0\\0\end{matrix}\right)$, $\left.\left|j\right.\right〉≐\left(\begin{matrix}0\\1\\0\end{matrix}\right)$, and $\left.\left|k\right.\right〉≐\left(\begin{matrix}0\\0\\1\end{matrix}\right)$. Compute the outer products $\left|\left.i\right〉\left〈i\right.\right|$, $\left|\left.j\right〉\left〈j\right.\right|$, and $\left|\left.k\right〉\left〈k\right.\right|$ in matrix form. Add the matrices to find the operator $\hat{I}=\left|\left.i\right〉\left〈i\right.\right|$+$\left|\left.j\right〉\left〈j\right.\right|+\left|\left.k\right〉\left〈k\right.\right|$ in this basis.
2. Use matrix multiplication to compute $\hat{I}\left|\left.F\right〉\right.$, where $\left|\left.F\right〉\right.≐\left(\begin{matrix}a\\b\\c\end{matrix}\right)$. $\hat{I}$ is called an identity operator. Describe the effect of operator $\hat{I}$ on $\left|\left.F\right〉\right.$ in a sentence.
3. In the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation, the normalized basis vectors $\left.\left|i'\right.\right⟩$, $\left.\left|j'\right.\right⟩$, and $\left.\left|k'\right.\right⟩$ are represented as $\left.\left|i'\right.\right〉≐\left(\begin{matrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{matrix}\right)$, $\left.\left|j'\right.\right〉≐\left(\begin{matrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{matrix}\right)$, and $\left.\left|k'\right.\right〉≐\left(\begin{matrix}0\\0\\1\end{matrix}\right)$ . Compute the outer products $\left|\left.i'\right〉\left〈i'\right.\right|$, $\left|\left.j'\right〉\left〈j'\right.\right|$, and $\left|\left.k'\right〉\left〈k'\right.\right|$ in matrix form. Add the matrices to find the operator $\hat{I}^{'}=\left|\left.i'\right〉\left〈i'\right.\right|$+$\left|\left.j'\right〉\left〈j'\right.\right|+\left|\left.k'\right〉\left〈k'\right.\right|$ in this basis.
4. Is there any similarity between the matrices $\hat{I}$ and $\hat{I}'$ you found in parts (1) and (3) by computing the outer product of each unit vector in a given basis with itself and then adding all of them? Explain.

In general, given an orthonormal basis (coordinates), if we compute the outer product of each unit vector with itself and then add all of them, we obtain the identity operator. This relation is often called the **completeness relation**. Since the identity operator does not change a vector $\left|\left.F\right〉\right.$, the completeness relation is extremely useful for decomposing a vector $\left|\left.F\right〉\right.$ into its components along the chosen basis vectors, e.g., $\left|\left.i\right〉\right., \left|\left.j\right〉\right., and \left|\left.k\right〉\right.$:

$\left|\left.F\right〉\right.=\hat{I}\left|\left.F\right〉\right.=(\left|\left.i\right〉\left〈i\right.\right|$+$\left|\left.j\right〉\left〈j\right.\right|+\left|\left.k\right〉\left〈k\right.\right|)\left|\left.F\right〉\right.=\left|\left.i\right〉\left〈i\right.\right|\left.F\right〉+\left|\left.j\right〉\left〈j\right.\right|\left.F\right〉+\left|\left.k\right〉\left〈k\right.\right|\left.F\right〉=a\left|\left.i\right〉\right.+b\left|\left.j\right〉\right.+c\left|\left.k\right〉\right.$ (11)

where $a, b, and c$ are the components of $\left|\left.F\right〉\right.$ along the basis vectors $\left|\left.i\right〉\right., \left|\left.j\right〉\right., and \left|\left.k\right〉\right.$ as given by Eq. (4).

The identity operator can also be useful to find the relationship between components of a generic vector in different bases.

* Suppose the force vector is represented as $\left|\left.F\right〉\right.=a\left|\left.i\right〉\right.+b\left.\left|j\right.\right〉+c\left.\left|k\right.\right〉$ in the {$\left.\left|i\right.\right⟩$, $\left.\left|j\right.\right⟩$, $\left.\left|k\right.\right⟩$} representation. Thus, $a=\left⟨F\right⟩$, $b=\left⟨F\right⟩$, $c=\left⟨F\right⟩$.
* We then choose a **new** basis {$\left.\left|i\right.'\right⟩$, $\left.\left|j\right.'\right⟩$, $\left.\left|k'\right.\right⟩$} (shown in Figure 3) such that the force vector is represented as $\left|\left.F\right〉\right.=a'\left|\left.i^{'}\right〉\right.+b'\left.\left|j^{'}\right.\right〉+c'\left.\left|k'\right.\right〉$ in the {$\left.\left|i\right.'\right⟩$, $\left.\left|j\right.'\right⟩$, $\left.\left|k'\right.\right⟩$} representation.

Then, $a'=\left⟨F\right⟩$, $b'=\left⟨F\right⟩$, $c'=\left⟨F\right⟩$.



Figure 3

* How can we find a relationship between the components $a, b, and c$ and the components $a^{'}$, $b^{'}$, and $c^{'}$?

One strategy, for example, to find $a$ in terms of $a^{'}$, $b^{'}$, and $c^{'}$ is to insert the identity operator in terms of the basis vectors $\left.\left|i\right.'\right⟩$, $\left.\left|j\right.'\right⟩$ and $\left.\left|k'\right.\right⟩$, e.g., $\hat{I}^{'}=\left|\left.i'\right〉\left〈i'\right.\right|$+$\left|\left.j'\right〉\left〈j'\right.\right|+\left|\left.k'\right〉\left〈k'\right.\right|$ as follows:

$a=\left⟨F\right⟩=\left⟨F\right⟩=\left.\left〈i\right.\right| (\left|\left.i'\right〉\left〈i'\right.\right|$+$\left|\left.j'\right〉\left〈j'\right.\right|+\left|\left.k'\right〉\left〈k'\right.\right|) \left.\left|F\right.\right〉=\left〈i\right.\left|\left.i'\right〉\left〈i'\right.\right|\left.F\right〉+\left〈i\right.\left|\left.j'\right〉\left〈j'\right.\right|\left.F\right〉+\left〈i\right.\left|\left.k'\right〉\left〈k'\right.\right|\left.F\right〉$

Using Eq. 8, $a=\left〈i\right.\left|\left.i'\right〉\left〈i'\right.\right|\left.F\right〉+\left〈i\right.\left|\left.j'\right〉\left〈j'\right.\right|\left.F\right〉+\left〈i\right.\left|\left.k'\right〉\left〈k'\right.\right|\left.F\right〉=\frac{1}{\sqrt{2}}(a^{'}-b^{'})$.

**Checkpoint 7**



1) Suppose in the{$\left.\left|i'\right.\right〉, \left.\left|j'\right.\right〉,\left.\left|k'\right.\right〉$} representation, the force vector $\left|\left.F\right〉\right.≐\left(\begin{matrix}a^{'}\\b^{'}\\c^{'}\end{matrix}\right)=a'\left|\left.i^{'}\right〉\right.+b'\left.\left|j'\right.\right〉+c\left.'\left|k'\right.\right〉$. Are the components of $\left|\left.F\right〉\right.$ the same as the components $a, b, and c$ along the basis vectors in the {$\left.\left|i\right.\right〉, \left.\left|j\right.\right〉, \left.\left|k\right.\right〉$} representation? Explain. Find a relation between $a^{'}$ and $a, b, and c$ if the basis vectors in the two representations are related by the figure shown above.

2) In the {$\left.\left|i\right.\right〉, \left.\left|j\right.\right〉,\left.\left|k\right.\right〉$} representation, the force vector is $\left|\left.F\right〉\right.≐\left(\begin{matrix}3 N\\7 N\\2 N\end{matrix}\right)=3N\left|\left.i\right〉\right.+7N\left|\left.j\right〉\right.+2N\left|\left.k\right〉\right.$. Consider the {$\left.\left|i'\right.\right〉, \left.\left|j'\right.\right〉,\left.\left|k'\right.\right〉$} representation in which the force is $\left|\left.F\right〉\right.=a'\left|\left.i'\right〉\right.+b'\left.\left|j'\right.\right〉+c\left.'\left|k'\right.\right〉$. What are the components $a', b', and c'$ of force $\left|\left.F\right〉\right.$ if the basis vectors in the two representations are related by Figure 3?

3) Consider the following statements from Student A and Student B.

Student A: In a given basis, vector $\left|\left.F\right〉\right.$ can be written as a column matrix, an outer product $\left|\left.F\right〉\right.\left〈\left.G\right|\right.$ as a square matrix, and a scalar product $\left⟨G\right⟩$ as a number.

Student B: I only agree with the first part. I don’t think that $\left|\left.F\right〉\right.\left〈\left.G\right|\right.$ is different from $\left⟨G\right⟩$. Both of these are operators.

Explain why you agree or disagree with Student A and Student B.

**Projection Operator**

Suppose we want to find the projection of a vector $\left|\left.F\right〉\right.=a\left|\left.i\right〉\right.+b\left.\left|j\right.\right〉+c\left.\left|k\right.\right〉$ onto one of the orthonormal basis vectors, e.g., $\left|\left.i\right〉\right., \left.\left|j\right.\right〉, or \left.\left|k\right.\right〉$. To project $\left|\left.F\right〉\right. $ onto one of the orthonormal basis vectors, we can act on the vector $\left|\left.F\right〉\right.$ with a projection operator corresponding to the basis vector. If we want to find the projection of $\left|\left.F\right〉\right.$ along the basis vector $\left|\left.i\right〉\right.$ (or $x$-coordinate), we can use the operator $\left|\left.i\right〉\left〈\left.i\right|\right.\right.$ acting on vector $\left|\left.F\right〉\right.$ to obtain

$\left|\left.i\right〉\left〈\left.i\right|\left.F\right〉\right.=a\right.\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.i\right〉+b\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.j\right〉+c\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.k\right〉=a\left|\left.i\right〉\right.$.

We can check that the matrix representation of the projection operator $\left|\left.i\right〉\left〈\left.i\right|\right.\right. $acting on the vector $\left|\left.F\right〉\right.$ will also give us the same result for the projection of $\left|\left.F\right〉\right.$ along $\left|\left.i\right〉\right..$

$$\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.F\right〉≐\left(\begin{matrix}1\\0\\0\end{matrix}\right)\left(\begin{matrix}1&0&0\end{matrix}\right)\left(\begin{matrix}a\\b\\c\end{matrix}\right)=\left(\begin{matrix}1&0&0\\0&0&0\\0&0&0\end{matrix}\right)\left(\begin{matrix}a\\b\\c\end{matrix}\right)=\left(\begin{matrix}a\\0\\0\end{matrix}\right)=a\left(\begin{matrix}1\\0\\0\end{matrix}\right)=a\left|\left.i\right〉\right.$$

The projection of $\left|\left.F\right〉\right.$ along a basis vector is also a vector.

**Checkpoint 8**:

1. Consider the following conversation between Student A and Student B:
* Student A: The projection operator acting on vector $\left|\left.F\right〉\right.$ is like the identity operator in that it returns the same vector $\left|\left.F\right〉\right.$ back along the direction of a basis vector.
* Student B: I disagree. The projection operator, e.g., $\left|\left.i\right〉\left〈\left.i\right|\right.\right.$, acting on vector $\left|\left.F\right〉\right.$ is not like the identity operator acting on vector $\left|\left.F\right〉\right.$. When the projection operator $\left|\left.i\right〉\left〈\left.i\right|\right.\right.$ acts on $\left|\left.F\right〉\right.$, it does not return the same vector $\left|\left.F\right〉\right.$. Rather, it returns a basis vector $\left.\left|i\right.\right〉$ multiplied by the component of $\left|\left.F\right〉\right.$ along that basis vector $\left.\left|i\right.\right〉$, e.g., $\left〈\left.i\right|\left.F\right〉\right.$.

With whom do you agree? Explain your reasoning.

1. Consider the conversation between Student A and Student B:
* Student A: In the {$\left.\left|i'\right.\right〉, \left.\left|j'\right.\right〉,\left.\left|k'\right.\right〉$} representation, the force vector $\left|\left.F\right〉\right.=a'\left|\left.i^{'}\right〉\right.+b'\left.\left|j^{'}\right.\right〉+c'\left.\left|k'\right.\right〉$. In the {$\left.\left|i\right.\right〉, \left.\left|j\right.\right〉,\left.\left|k\right.\right〉$} representation, $\left|\left.F\right〉\right.=a\left|\left.i\right〉\right.+b\left.\left|j\right.\right〉+c\left.\left|k\right.\right〉$. Suppose the basis vectors in the two representations are related as in Figure 3. To determine if $\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.F\right〉=\left.\left|i'\right.\right〉\left〈\left.i'\right|\left.F\right〉\right.$ , we must calculate the inner products $\left⟨F\right⟩$ and $\left⟨F\right⟩$.
* Student B: I disagree with you. If the basis vectors in the two representations are related as in Figure 3, it is not possible that $\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.F\right〉=\left.\left|i'\right.\right〉\left〈\left.i'\right|\left.F\right〉\right.$ because $\left.\left|i\right.\right⟩$ and $\left.\left|i\right.'\right⟩$ are basis vectors which point in different directions. Thus, there is no need to calculate the inner products $\left⟨F\right⟩$ and $\left⟨F\right⟩$ to determine if $\left|\left.i\right〉\left〈\left.i\right|\right.\right.\left.F\right〉=\left.\left|i'\right.\right〉\left〈\left.i'\right|\left.F\right〉\right.$.

With whom do you agree? Explain your reasoning.