Uncertainty Principle Tutorial part II

- In quantum mechanics, for any observable A or B, there is a corresponding Hermitian operator $\hat{A}$ or $\hat{B}$ in the Hilbert space in which the state of the system lies.
- If two operators are incompatible, measuring an observable corresponding to one operator will affect the probability of measuring the observable corresponding to the other operator.
- We can find a complete set of simultaneous eigenstates for compatible operators.
- We cannot find a complete set of simultaneous eigenstates for incompatible operators.
- $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right)^2$ is the generalized uncertainty principle that relates the product of the variances of observables A or B.

The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined by $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. If $\hat{A}$ and $\hat{B}$ commute with each other, i.e., $[\hat{A},\hat{B}] = 0$, we call them compatible operators. Otherwise, $\hat{A}$ and $\hat{B}$ are called incompatible operators. Assume all the operators in this tutorial correspond to physical observables and have non-degenerate eigenstates unless the degeneracy is explicitly mentioned.

Case I: $\hat{A}$ and $\hat{B}$ commute

Suppose Hermitian operators $\hat{A}$ and $\hat{B}$ commute with each other, answer questions 1 to 3.

1. If $|\alpha\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue α, is $\hat{B}|\alpha\rangle$ an eigenstate of $\hat{A}$? If so, what is the eigenvalue of $\hat{A}$ corresponding to the state $\hat{B}|\alpha\rangle$? Explain. (Hint: $[\hat{A},\hat{B}]|\alpha\rangle = 0$, so $\hat{A}\hat{B}|\alpha\rangle = \hat{B}\hat{A}|\alpha\rangle = \hat{B}\alpha|\alpha\rangle$. Remember that $\hat{A}$ and $\hat{B}$ have only <u>**non-degenerate**</u> eigenstates.)

2. If $|\alpha\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue α, is $|\alpha\rangle$ an eigenstate of $\hat{B}$? Explain. (Hint: $|\alpha\rangle$ and $\hat{B}|\alpha\rangle$ are both eigenstates of $\hat{A}$ with the same eigenvalue α. Are they proportional to each other, i.e., $\hat{B}|\alpha\rangle = \beta|\alpha\rangle$ where β is a constant? Remember that $\hat{A}$ and $\hat{B}$ have only <u>**non-degenerate**</u> eigenstates.)

3. Consider the following conversation between Andy and Caroline:

Caroline: Now I understand that any eigenstate $|\psi_n\rangle$ of the operator $\hat{A}$ must be an eigenstate of the operator $\hat{B}$ if $\hat{A}$ and $\hat{B}$ commute with each other. But how can we know that $|\psi_n\rangle$ also form a complete set of eigenstates for $\hat{B}$?

Andy: When $\hat{A}$ and $\hat{B}$ have only non-degenerate eigenstates, by following the same method as in the previous question we can show that any eigenstate $|\psi_n\rangle$ of the operator $\hat{A}$ must be an eigenstate of the operator $\hat{B}$. Thus, we can prove that any eigenstate of the operator $\hat{B}$ must also be an eigenstate of $\hat{A}$ if their eigenvalue spectra are non-degenerate. Therefore, the complete set of eigenstates $|\psi_n\rangle$ are shared by both the compatible operators $\hat{A}$ and $\hat{B}$.

Do you agree with Andy? Explain.

4. Suppose operators $\hat{A}$ and $\hat{B}$ <u>commute</u> with each other. $|\chi\rangle$ is a common eigenstate of both $\hat{A}$ and $\hat{B}$ with eigenvalues α and β respectively ($\hat{A}|\chi\rangle = \alpha|\chi\rangle$, $\hat{B}|\chi\rangle = \beta|\chi\rangle$). At time t=0, the state of the quantum system is $|\psi\rangle$. If $\hat{A}$ and $\hat{B}$ have only <u>**non-degenerate**</u> eigenstates, answer the following questions and express the probabilities in Dirac notation.

   (a) What is the probability of obtaining the value α if we measure the observable A (corresponding to the operator $\hat{A}$) in the state $|\psi\rangle$ at t=0?

   (b) Suppose the wavefunction collapses to $|\chi\rangle$ after your measurement of A at time t=0. If you measure the observable B (corresponding to the operator $\hat{B}$) immediately after the measurement of A at time t=0, what value will you get? What is the overall probability (compared to the initial state $|\psi\rangle$) of obtaining this value?

(c) If you directly measure the observable B in the state $|\psi\rangle$ at time t=0 without measuring A first, what is the probability of obtaining the value β? Is this result the same as your answer in the previous problem (question (b))? Explain.

(d) When $\hat{A}$ and $\hat{B}$ are compatible operators with non-degenerate eigenstates and the initial state of the system is $|\psi\rangle$, does the probability of obtaining β depend on whether we measure A first? Explain.

Case II: $\hat{A}$ and $\hat{B}$ do not commute

5. Suppose operators $\hat{A}$ and $\hat{B}$ **do not** commute with each other and $[\hat{A},\hat{B}]=q$ is a non-zero constant. If $|\alpha\rangle$ is an eigenstate of $\hat{A}$ with nonzero eigenvalue α, could $|\alpha\rangle$ also be an eigenstate of $\hat{B}$? Explain. (Hint: Assume $\hat{B}|\alpha\rangle=\beta|\alpha\rangle$ and calculate the value of $[\hat{A},\hat{B}]|\alpha\rangle$ to show that the assumption is incorrect if $[\hat{A},\hat{B}]=q$.)

6. Suppose operators $\hat{A}$ and $\hat{B}$ **do not** commute with each ($[\hat{A},\hat{B}]\neq 0$ but the commutator can either be a constant value or an operator). $|\psi_n\rangle$ form a complete set of eigenstates of $\hat{A}$ with eigenvalues α_n. Could $|\psi_n\rangle$ also be a complete set of eigenstates of $\hat{B}$? Explain. (Hint: For any state $|\Psi\rangle=\sum_n c_n|\psi_n\rangle$, show that $[\hat{A},\hat{B}]|\Psi\rangle=0$ if $|\psi_n\rangle$ form a complete set of eigenstates for both $\hat{A}$ and $\hat{B}$. But this assumption cannot be true if $[\hat{A},\hat{B}]\neq 0$.)

7. Suppose operators $\hat{A}$ and $\hat{B}$ **do not** commute with each other. $|\alpha\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue α and $|\beta\rangle$ is an eigenstate of $\hat{B}$ with eigenvalue β ($\hat{A}|\alpha\rangle = \alpha|\alpha\rangle$, $\hat{B}|\beta\rangle = \beta|\beta\rangle$). At time *t*=0, the state of the quantum system is $|\psi\rangle$. Answer the following questions and express the probability in Dirac notation.

(a) At time *t*=0, if we measure the observable A (corresponding to the operator $\hat{A}$) in the state $|\psi\rangle$, what is the probability of obtaining the value α? Explain.

(b) Suppose the measurement of A collapses the wavefunction into the state $|\alpha\rangle$. If we measure the observable B (corresponding to the operator $\hat{B}$) immediately after the measurement of A at time *t*=0, what is the probability (compared to the state $|\alpha\rangle$) of obtaining the value β? Explain.

(c) In the previous problem (question 7.b), if we don't know the outcome after the measurement of A, what is the overall probability (compared to the initial state $|\psi\rangle$) of obtaining the value β if we measure B after the measurement of A? Explain. (Hint: suppose the complete set of eigenstates of the operator $\hat{A}$ are $|\psi_n\rangle$. Calculate the probability of the initial state $|\psi\rangle$ collapsing to $|\psi_n\rangle$ first when A is measured and then calculate the probability of the state $|\psi_n\rangle$ collapsing to $|\beta\rangle$ when B is measured.)

(d) If we directly measure the observable B in the state $|\psi\rangle$ at time t=0 without measuring A first, what is the probability of obtaining the value β? Is this result the same as your answer in the previous problem (question (c))? Explain.

(e) When $\hat{A}$ and $\hat{B}$ are incompatible operators with non-degenerate eigenstates and the initial state of the system is $|\psi\rangle$, does the probability of obtaining β depend on whether we measure A first? Explain.

8. (a) Consider the following conversation between Andy and Caroline:

Caroline: I don't understand the answer to the previous problem. How can the measurement of A affect the overall probability of obtaining the value β of the observable B?

Andy: Suppose the eigenstates of the operator $\hat{A}$ are $|\psi_n\rangle$. If we measure A first, the probability of the initial state collapsing to $|\psi_n\rangle$ is $|\langle\psi_n|\psi\rangle|^2$. The preceding measurement of B following the measurement of A has a probability $|\langle\beta|\psi_n\rangle|^2$ of obtaining the value β. Therefore, the overall probability of obtaining β is $\sum_n |\langle\psi_n|\psi\rangle|^2 |\langle\beta|\psi_n\rangle|^2 = \sum_n |\langle\psi_n|\psi\rangle\langle\beta|\psi_n\rangle|^2$ if we measure the observable A before measuring B in the initial state $|\psi\rangle$.

Caroline : But if we measure B in the state $|\psi\rangle$ directly, the probability of obtaining β is $|\langle\beta|\psi\rangle|^2$. Since $|\psi_n\rangle$ form a complete set of vectors, $\sum_n |\psi_n\rangle\langle\psi_n| = 1$, we know $\langle\beta|\psi\rangle = \sum_n \langle\beta|\psi_n\rangle\langle\psi_n|\psi\rangle$. Isn't this probability $|\langle\beta|\psi\rangle|^2$ the same as the overall probability of getting β if we measure B after the measurement of A?

Andy: No. Because the probability of measuring β directly in the state $|\psi\rangle$ is $|\langle\beta|\psi\rangle|^2 = \left|\sum_n \langle\beta|\psi_n\rangle\langle\psi_n|\psi\rangle\right|^2$. And the probability of obtaining β if we measure B after the measurement of A is $\sum_n |\langle\beta|\psi_n\rangle\langle\psi_n|\psi\rangle|^2$. Generally, $\left|\sum_n a_n\right|^2 \neq \sum_n |a_n|^2$.

Do you agree with Andy? Explain whether the probabilities of obtaining the eigenvalue $\hbar/2$ are the same in the situation where the incompatible operators are $\hat{A} = \hat{S}_x$, $\hat{B} = \hat{S}_z$ and the initial state is $|\psi\rangle = |\uparrow_z\rangle$ for the following cases.

(i) B is measured directly in the state $|\psi\rangle = |\uparrow_z\rangle$.

(ii) B is measured after the measurement of A in the state $|\psi\rangle = |\uparrow_z\rangle$.

(b) Consider the following conversation between Andy and Caroline:

Caroline: Now I understand that when the operators do not commute, the probability of measuring B depends on whether A is measured before in the state $|\psi\rangle$. However, if the operators $\hat{A}$ and $\hat{B}$ commute with each other, why is the probability of measuring B the same in the state $|\psi\rangle$ whether it is measured directly or after the measurement of A?

Andy: When the Hermitian operators $\hat{A}$ and $\hat{B}$ are compatible, they have a complete set of simultaneous eigenstates $|\psi_n\rangle$. Let's consider the probability of obtaining a particular eigenvalue β when measuring the observable B. Suppose the eigenstate corresponding to the eigenvalue β is $|\beta\rangle = |\psi_1\rangle$. We know $\langle\beta|\psi_n\rangle$ equals zero for all the $|\psi_n\rangle$ except for $n=1$. So $\left|\sum_n \langle\beta|\psi_n\rangle\langle\psi_n|\psi\rangle\right|^2 = |\langle\psi_1|\psi\rangle|^2 = \sum_n |\langle\beta|\psi_n\rangle\langle\psi_n|\psi\rangle|^2$. That's why the probability does not change when $\hat{A}$ and $\hat{B}$ commute with each other as we have seen in question 4(d) .

Do you agree with Andy? Explain.

9. Which one of the following equations correctly represents the commutation relation between the *x*-component and *z*-component of the angular momentum operators $\hat{L}_x$ and $\hat{L}_y$?

A. $[\hat{L}_x, \hat{L}_y] = 0$

B. $[\hat{L}_x, \hat{L}_y] = \hat{L}_z$

C. $[\hat{L}_x, \hat{L}_y] = \hbar\hat{L}_z$

D. $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$

10. Can we measure $\hat{L}_x$ and $\hat{L}_y$ simultaneously in a given quantum state? Explain.

11. According to your answer to the previous question, can you measure the magnitude and direction of the angular momentum operator $\hat{\vec{L}} = \hat{L}_x\vec{i} + \hat{L}_y\vec{j} + \hat{L}_z\vec{k}$ simultaneously in a given quantum state? Explain.

12. (a) Consider the following conversation between Andy and Caroline.

Caroline: Does the state of the system determine whether you can measure two observables simultaneously?

Andy: No. Whether you can measure two observables simultaneously only depends on the commutation relation between the corresponding operators. Only when the operators are compatible can we measure the corresponding observables at the same time (both observables can be well-defined in a given quantum state).

Do you agree with Andy? Explain.

(b) Consider the following conversation between Andy and Caroline.

Caroline: But what if the expectation value of the commutator $\langle [\hat{A}, \hat{B}] \rangle$ is zero in a particular state $|\Psi\rangle$. I know for any operator $\hat{A}$ and $\hat{B}$, the generalized uncertainty principle gives $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$. Does that mean that the operators $\hat{A}$ and $\hat{B}$ can be measured simultaneously in the state $|\Psi\rangle$ when $\langle [\hat{A}, \hat{B}] \rangle = 0$?

Andy. No. The uncertainty principle is an inequality involving $\sigma_A^2 \sigma_B^2$. It says that $\sigma_A^2 \sigma_B^2$ has a value *larger* than or equal to $\left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$. In fact, $\sigma_A^2 \sigma_B^2$ is usually larger than $\left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$. For example, we know that the commutator of the *x*-component and the *y*-component of the angular momentum operator $\hat{L}_x$ and $\hat{L}_y$ is $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$. It is possible that the **expectation value** of $[\hat{L}_x, \hat{L}_y]$ equals zero if the state of the system is $|\uparrow\rangle_x$ or $|\uparrow\rangle_y$. But since $\hat{L}_x$ and $\hat{L}_y$ do not commute with each other, i.e., $[\hat{L}_x, \hat{L}_y] \neq 0$, you cannot measure the *x*-component and the *y*-component of the angular momentum simultaneously in a given quantum state.

Do you agree with Andy? Explain.

Case III: Operators that commute but have some degenerate eigenstates

The energy of a hydrogen atom is $E_n = \frac{-13.6eV}{n^2}$ *(n=1, 2, 3, …) when we only consider the Coulomb interaction. The angular momentum eigenstates for the electron in a Hydrogen atom with energy* E_n *can be represented by* $|\ell, m\rangle$ *where* $\ell = 0,1,2,\ldots,n-1$ *is the quantum number for the square of the angular momentum operator* $\hat{L}^2$ *and* $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$ *is the quantum number for the z-component of angular momentum* $\hat{L}_z$. *Answer the following questions.*

13. Which one of the following equations correctly represents the commutation relation between the operators $\hat{L}^2$ and $\hat{L}_z$?

A. $[\hat{L}^2, \hat{L}_z] = 0$

B. $[\hat{L}^2, \hat{L}_z] = \hbar$

C. $[\hat{L}^2, \hat{L}_z] = i\hbar$

D. $[\hat{L}^2, \hat{L}_z] = L_x + iL_y$

14. Consider the following conversation between Andy and Caroline.

Caroline: Since the square of the angular momentum operator $\hat{L}^2$ and the *z*-component of the angular momentum operator $\hat{L}_z$ commute with each other, we can infer the value of L_z after the measurement of L^2, right?

Andy: No. $\hat{L}^2$ has degenerate eigenvalues $\ell(\ell+1)\hbar^2$ where ℓ is the quantum number corresponding to the square of the angular momentum. A given value of ℓ corresponds to $2\ell+1$ eigenstates $|\ell, m\rangle$ where $m = -\ell, -\ell+1, ..., \ell-1, \ell$ is the quantum number for the *z*-component of angular momentum. For example, consider the case when the initial state of the electron in the ground state of the Hydrogen atom is $|\psi\rangle = \frac{1}{2}(|0,0\rangle + |1,0\rangle + |1,-1\rangle + |1,1\rangle)$. If you measure L^2 in the state $|\psi\rangle$ and obtain $2\hbar^2$ which corresponds to the quantum number $\ell = 1$, the angular momentum state of the electron would collapse to a normalized superposition of $|1,0\rangle$ and $|1,\pm 1\rangle$, i.e., $|\psi'\rangle = \frac{1}{\sqrt{3}}(|1,0\rangle + |1,-1\rangle + |1,1\rangle)$, which is an eigenstate of $\hat{L}^2$. However, in this state $|\psi'\rangle$, the *z*-component of angular momentum L_z does not have a definite value so it is not an eigenstates of $\hat{L}_z$.

Caroline: But what if the measurement of L^2 in the state $|\psi\rangle$ collapses the wavefunction to $|0,0\rangle$? Can we infer the value of both L^2 and L_z in this case?

Andy: Yes. If the state of the system collapse to $|0,0\rangle$ after the measurement of L^2, the *z*-component of the angular momentum will have a definite value.

Do you agree with Andy? Explain.

15. Suppose the initial state of the electron in the ground state of the Hydrogen atom is $|\psi\rangle = \frac{1}{2}\left(|0,0\rangle + |1,0\rangle + |1,-1\rangle + |1,1\rangle\right)$. Use the previous question (question 14) as a guide to answer questions 15(a) and 15(b).

(a) If you measure the *z*-component of the angular momentum L_z in the state $|\psi\rangle$ and obtain the eigenvalue 0 (corresponding to the quantum number $m = 0$), does the electron have a definite value of the square of angular momentum L^2 after your measurement? Explain.

(b) If you measure the *z*-component of the angular momentum L_z for the state $|\psi\rangle$ and obtain $\hbar$ (corresponding to the quantum number $m = 1$), does the electron have a definite value of the square of the angular momentum L^2 after your measurement? Explain.

16. We can find a complete set of simultaneous eigenstates for $\hat{L}^2$ and $\hat{L}_z$ (because they commute although the eigenvalue spectra of $\hat{L}^2$ has a degeneracy).

Caroline: Now I can see that we may or may not infer the value of L^2 after the measurement of L_z. Whether L^2 has a definite value after the measurement of L_z depends on what state the system collapses into. But does it mean that we cannot measure L^2 and L_z simultaneously in any state?

Andy: No. The observables L^2 and L_z can be measured simultaneously because the operators $\hat{L}^2$ and $\hat{L}_z$ commute with each other and they share a complete set of eigenstates $|\ell, m\rangle$. It is true that some superposition of the basis vectors $|\ell, m\rangle$ can be an eigenstate of $\hat{L}_z$ but not of $\hat{L}^2$, and some other superposition of $|\ell, m\rangle$ can be an eigenstate of $\hat{L}^2$ but not of $\hat{L}_z$. However, the probability of obtaining a particular eigenvalue of the operator $\hat{L}^2$ does not depend on whether we measure L_z first, and vice versa. For example, if we measure L^2 directly for the state $|\psi\rangle = \frac{1}{2}\left(|0,0\rangle + |1,0\rangle + |1,-1\rangle + |1,1\rangle\right)$, the probability of obtaining $\ell = 0$ is $\frac{1}{4}$. If we measure L_z first, the probabilities of obtaining $m = 0$, $m = 1$ and $m = -1$ are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ respectively. Only when the state of the system collapses to $\frac{|0,0\rangle + |1,0\rangle}{\sqrt{2}}$ after the measurement of L_z, i.e., $m = 0$, can we obtain the value $\ell = 0$ if we measure L^2 following the measurement of L_z. Therefore, if we measure L_z first for the state $|\psi\rangle$ and then measure L^2, the probability of obtaining $\ell = 0$ is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, which is the same as measuring L^2 directly for the state $|\psi\rangle$.

Do you agree with Andy? Explain.