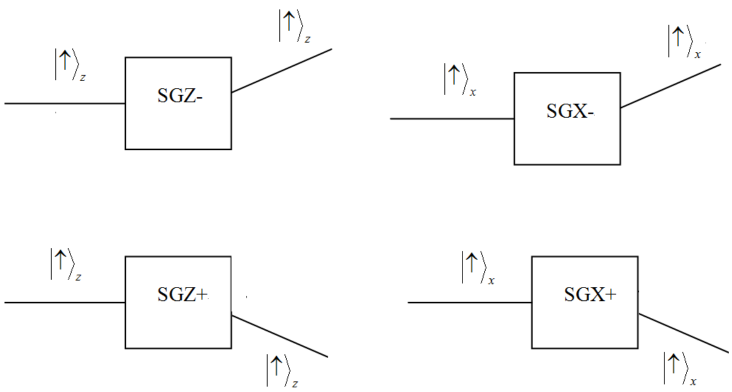
Tutorial for Stern-Gerlach Experiment

The figure below shows the pictorial representations used for a Stern-Gerlach apparatus (SGA). If an atom with state $\left|\uparrow\right\rangle_z$ (or $\left|\downarrow\right\rangle_z$) passes through a Stern-Gerlach apparatus with the field gradient in the negative *z*-direction (SGZ-), it will be deflected in the +z (or -*z*) direction. If an atom with state $\left|\uparrow\right\rangle_z$ (or $\left|\downarrow\right\rangle_z$) passes through a Stern-Gerlach apparatus with the field gradient in the positive *z*-direction (SGZ+), it will be deflected in the -*z* (or +*z*) direction. Similarly, if an atom with state $\left|\uparrow\right\rangle_x$ passes through SGX- (or SGX+), it will be deflected in the +*x* (or -*x*) direction. The figures below show examples of deflections through the SGX and SGZ in the plane of the paper. However, note that the deflection through a SGX will be in a plane perpendicular to the deflection through an SGZ. This actual three-dimensional nature should be kept in mind in answering the questions.



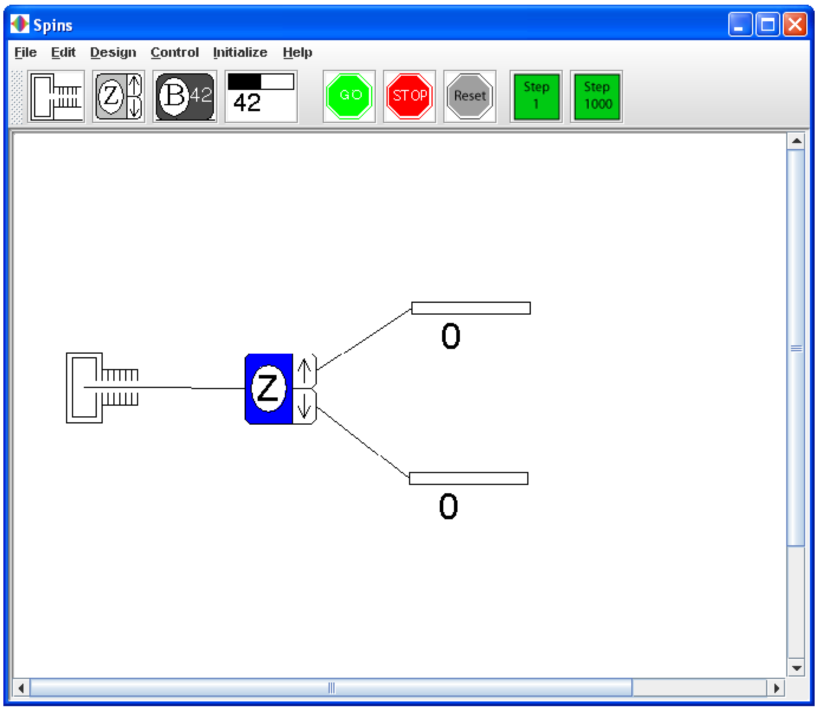
In this tutorial, we will learn about the basics of quantum mechanics via Stern-Gerlach experiment and use simulations to check the results of Stern-Gerlach experiment after making predictions. Let's do some practice first.

Prediction:

A beam of atoms in the initial state $|\uparrow\rangle_z$ passes the SGZ- (magnetic field gradient in the –z direction). Two detectors are placed after the SGZ- to count the atoms coming out of the upper and lower channel. What is the probability that each detector clicks when an atom passes?

Simulation:

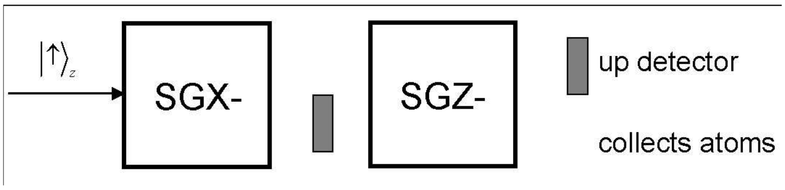
Double click the green arrow "z-up pass SGZ-" on the left column to check your answer. On the top of the simulation window, click the green button "GO" to start the simulation and click the red button "STOP" to pause the simulation. The "RESET" button clears all the detector counts to zero. The buttons "STEP1"/"STEP1000" send 1 or 1000 particles, respectively, through the SGA (Stern-Gerlach apparatus).



Now let's get started with the tutorial. Note that for all the SGAs used in the simulation, the magnetic field gradient is always in the negative direction. The gradient directions of SGX, SGY and SGZ shown in the simulation are along –x, –y and –z axes. So they will deflect the spin-up state to the upper channel and the spin-down state to the lower channel.

First predict the answers to the following problem. After that, the tutorial will provide systematic guidance in solving this problem.

You send silver atoms in an initial spin state $\left|\uparrow\right\rangle_z$ one at a time through two SGAs with magnetic field gradients as shown below. Suitable detectors are placed as shown. One detector is between the two SGAs (in the lower channel) and the other after both SGAs (in the upper channel). What is the probability that a given single atom will cause the "up" detector to click after passing through this system of two SGAs? What is the spin state of the atoms collected in the lower channel after SGZ-?



Step1: Write the initial state in a basis most suitable for analyzing the effects of passing through SGX-. [Hint: The time evolution of a system is convenient to analyze choosing the energy eigenstates as the basis vectors. If the Hamiltonian $\hat{H}$ commutes with the $\hat{S}_x$, the energy eigenstates are $\left|\uparrow\right\rangle_x$ and $\left|\downarrow\right\rangle_x$].

1. Which one of the following gives the correct relationship between the normalized eigenstates of $\hat{S}_z$ and $\hat{S}_x$.

A. $\left|\uparrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_x + i\left|\downarrow\right\rangle_x\right)$, $\left|\downarrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_x - i\left|\downarrow\right\rangle_x\right)$

B. $\left|\uparrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_x + \left|\downarrow\right\rangle_x\right)$, $\left|\downarrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_x - \left|\downarrow\right\rangle_x\right)$

C. $\left|\uparrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(a\left|\uparrow\right\rangle_x + b\left|\downarrow\right\rangle_x\right)$, $\left|\downarrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(a\left|\uparrow\right\rangle_x - b\left|\downarrow\right\rangle_x\right)$, where a and b can be any complex numbers that satisfy $|a|^2 + |b|^2 = 1$

D. $\left|\uparrow\right\rangle_z = \left|\downarrow\right\rangle_x$, $\left|\downarrow\right\rangle_z = \left|\uparrow\right\rangle_x$

2. John sends silver atoms in the $|\uparrow\rangle_z$ state through an SGX-. He places a "down" detector to block some silver atoms and collects the atoms coming out in the "upper channel". Which one of the following normalized spin states has John prepared in the "upper channel"? Think about how you can use the SGAs to check the state. Draw a figure below and explain. *Hint: if all of the atoms passing through an SGX are collected by the upper (or lower) detector, the spin state of the atoms is purely* $|\uparrow\rangle_x$ *(or* $|\downarrow\rangle_x$*).*

A. $|\uparrow\rangle_z$

B. $|\uparrow\rangle_x$

C. $\frac{1}{2}|\uparrow\rangle_z$

D. $\frac{1}{\sqrt{2}}|\uparrow\rangle_z$

☆Simulation: Now use the two simulations "z-up pass SGX-1" and "z-up pass SGX-2" to check your answers. In the first simulation, the atoms prepared in the "upper channel" passed through a SGZ-. In the second simulation, the atoms prepared in the upper channel passed through a SGX-. Is your prediction in *question 2* consistent with the observations in these simulations? If not, reconcile the difference.

Step2: Find the fraction of atoms that would pass through the second SGA (which were not absorbed by the first detector).

3. In *question 2*, what is the probability of the "down" detector clicking when John sends a silver atom? Does this probability depend on how much time the atom has stayed in the non-uniform magnetic field so long as the detectors are placed in appropriate locations after the SGA? Explain.

A. 1

B. 0

C. 0.5

D. It is between 0 and 0.5, but the exact probability cannot be inferred from the given information.

☆Simulation: After you have predicted the answer to the previous question, click on the simulation "z-up pass SGX-" to check your answers. In this simulation, we use detectors in both channels to estimate the probability. But if you want to prepare the atoms in $|\uparrow\rangle_x$ or $|\downarrow\rangle_x$ state, then you should use only one detector to block the unwanted component.

4. In *question 2*, if John measures $\hat{S}_x$ for the atoms he prepared in the "upper channel", what is the probability of measuring $+\frac{\hbar}{2}$?

A. 1
B. 0
C. 0.5
D. It is between 0 and 0.5, but the exact probability cannot be inferred from the given information.

☆Simulation: Now you can use "z-up pass SGX-2" to check your answers. Explain how your observation is consistent with your prediction. If it is not consistent, reconcile the difference.

In questions 1 to 4, you have learned the relationship between the eigenstates of $\hat{S}_z$ and $\hat{S}_x$. Now apply similar ideas to $\hat{S}_z$ and $\hat{S}_y$. Answer questions 5 to 8.

5. Which one of the following gives the correct relationship between the normalized eigenstates of $\hat{S}_z$ and $\hat{S}_y$.

A. $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_y + i|\downarrow\rangle_y\right)$, $|\downarrow\rangle_z = \frac{1}{\sqrt{2}}\left(i|\uparrow\rangle_y - |\downarrow\rangle_y\right)$

B. $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_y - i|\downarrow\rangle_y\right)$, $|\downarrow\rangle_z = \frac{-i}{\sqrt{2}}\left(|\uparrow\rangle_y + i|\downarrow\rangle_y\right)$

C. $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(a|\uparrow\rangle_x + b|\downarrow\rangle_x\right)$, $|\downarrow\rangle_z = \frac{1}{\sqrt{2}}\left(a|\uparrow\rangle_x - b|\downarrow\rangle_x\right)$, where a and b can be any complex numbers that satisfy $|a|^2 + |b|^2 = 1$

D. $|\uparrow\rangle_z = |\downarrow\rangle_y$, $|\downarrow\rangle_z = |\uparrow\rangle_y$

E. None of the above

6. Check that ${}_z\langle\uparrow|\downarrow\rangle_z = 0$ by expressing $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ in the $\hat{S}_y$ basis as above.

7. John sends silver atoms in the $|\uparrow\rangle_z$ state through a SGY-. He places a "down" detector to block some silver atoms and collects the atoms coming out in the "upper channel". What is the probability of the "down" detector clicking for each atom sent when John sends the silver atoms?

A. 1

B. 0

C. 0.5

D. It is between 0 and 0.5, but the exact probability cannot be inferred from the given information.

☆Simulation: Now you can click on the simulation "z-up pass SGY-" to check your answers. Explain how your observation is consistent with your prediction. If it is not consistent, reconcile the difference.

8. In the previous experiment, which one of the following normalized spin states has John prepared in the "upper channel"?

A. $|\uparrow\rangle_z$

B. $|\uparrow\rangle_y$

C. $\frac{1}{2}|\uparrow\rangle_y$

D. John has not prepared anything. Everything gets blocked by the "down" detector.

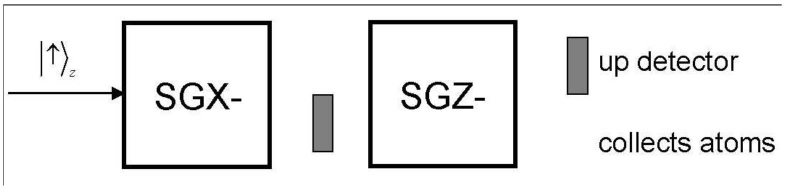
☆Simulation: Now try two simulations "z-up pass SGY-1" and "z-up pass SGY-2" to check your answers. In the first simulation, the atoms prepared in the "upper channel" passed through a SGZ-. In the second simulation, the atoms prepared in the upper channel passed through a SGX-. Based upon the observations in the two simulations, is your prediction in *question 8* consistent with the simulation?

Step3: As in Step 1, write the spin state of atoms before SGZ in a proper basis which helps to analyze the time evolution in SGZ.

9. The "down" detector between SGX and SGZ will collapse the state of the silver atoms. If the detector clicks, the atom gets absorbed by the detector. If the detector does not click, write down the spin state after passing through the SGX right before entering SGZ. Express this spin state in a basis that is most suitable for determining the time evolution after the atoms have passed through the SGZ.

Now let's solve the problem given at the beginning of this tutorial.

10. You send silver atoms in an initial spin state $\left|\uparrow\right\rangle_z$ one at a time through two SGAs with magnetic field gradients as shown below. Suitable detectors are placed as shown. One detector is between the two SGAs (in the lower channel) and the other after both SGAs (in the upper channel). What is the probability that a given single atom will cause the "up" detector to click after passing through this system of two SGAs?



A. 1
B. 0
C. 0.5
D. 0.25

☆Simulation : Now you can click the simulation "z-up pass SGX-1" to check your answer. Note that instead of collecting atoms, we have put detectors in both the upper and lower channels at the end to estimate the probability. Explain whether your prediction is consistent with the observation.

11. Consider the following conversation between Andy and Caroline:

Andy: I don't understand the answer to the previous question (*question 10*).

Caroline: When an atom in the state $\left|\uparrow\right\rangle_z = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_x + \left|\downarrow\right\rangle_x\right)$ passes through the SGX, each eigenstate of $\hat{S}_x$ gets spatially separated. If the detector between SGX and SGZ does not click, the state of that silver atom must have collapsed to $\left|\uparrow\right\rangle_x$. Since the atom in this state passes through SGZ next, we must write $\left|\uparrow\right\rangle_x = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_z + \left|\downarrow\right\rangle_z\right)$. You can see that $\left|\uparrow\right\rangle_x$ is a superposition of the eigenstates of $\hat{S}_z$ with equal weight to $\left|\uparrow\right\rangle_z$ and $\left|\downarrow\right\rangle_z$. This helps us find the probability that the second detector clicks.

Andy: Is an eigenstate of any one component of spin, say $\left|\uparrow\right\rangle_y$ for $\hat{S}_y$, a superposition of the eigenstates of any of the other two components, say $\hat{S}_z$ or $\hat{S}_x$, with equal weight?

Caroline: Yes. There may be a phase factor such as "i" (where $i = \sqrt{-1}$) when you write

$|\uparrow\rangle_y$ in terms of the superposition of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ but the probability is the same for both eigenstates of $\hat{S}_z$.

Do you agree with Caroline? Explain.

12. In the previous experiment (*question 10*), you collect the silver atoms that are not blocked at the end after they have passed through both SGAs. Which one of the following is the spin state of the silver atom you collect at the end in the lower channel?

A. $|\uparrow\rangle_z$

B. $|\downarrow\rangle_z$

C. $|\downarrow\rangle_x$

D. You do not collect anything because all atoms passing through the second SGA are blocked by the detector

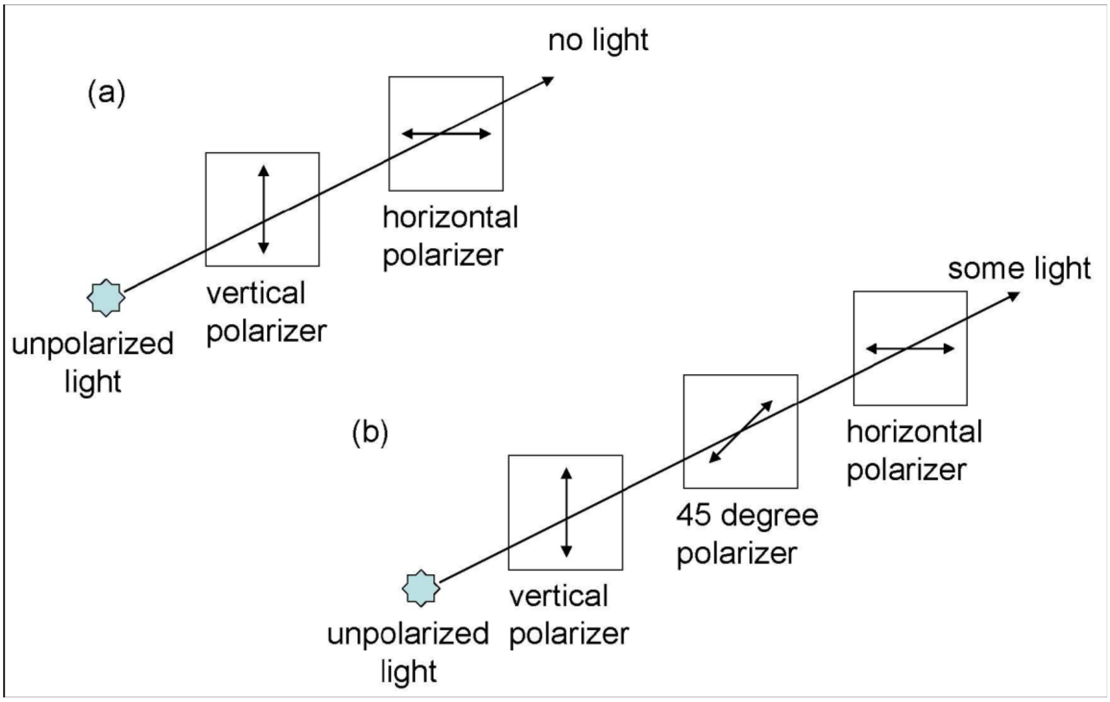
☆Simulation : Now you can click on the simulation "z-up pass SGX- and SGZ-" to check your answers. Explain any discrepancy between your prediction and observation. Note that the SGZ is only inserted to check the final state.

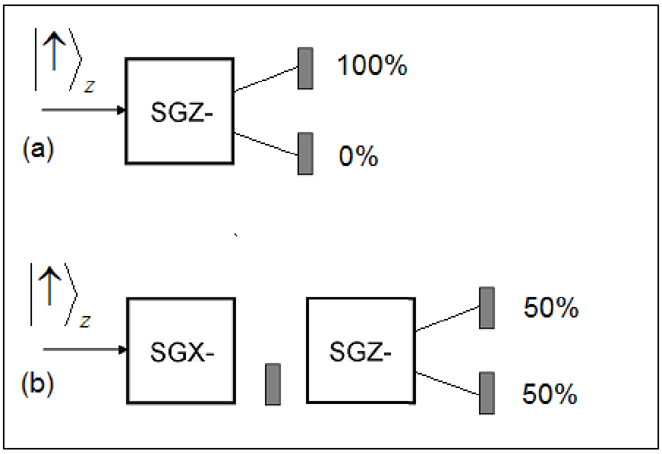
13. Consider the following conversation between Andy and Caroline:

Andy: There must be something wrong with the answer to the previous question (*question 12*). How can the $|\uparrow\rangle_z$ that we inputted give $|\downarrow\rangle_z$ on the way out?

Caroline: I disagree. If you let atoms in the state $|\uparrow\rangle_z$ pass through SGZ only, you will never obtain $|\downarrow\rangle_z$ on the way out. However, $|\downarrow\rangle_z$ is obtained in the above experiment because we have inserted SGX at an intermediate stage. Think of the analogy with vertically polarized light passing directly through a horizontal polarizer vs. passing first through a polarizer at 45^{o} followed by a horizontal polarizer. There is no light at the output if vertically polarized light passes directly through a horizontal polarizer. On the other hand, if the polarizer at 45^{o} is present, light becomes polarized at 45^{o} after the 45^{o} polarizer which is a linear superposition of horizontal and vertical polarization. Therefore, some light comes out

through the horizontal polarizer placed after the 45° polarizer.
Do you agree with Caroline? Explain. Also, comment on how good is the analogy between the spin-1/2 state of the atoms and the polarization state of photons.





Now consider the following problem. Questions 14 to 16 provide the steps to solve this problem.
Consider two situations as below.

Situation 1: The beam of atoms is in the pure state $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$.

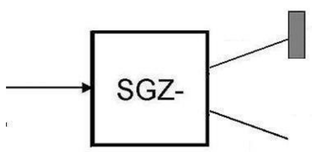
Situation 2: The beam of atoms is an unpolarized mixture, half of which is $|\uparrow\rangle_z$ ***and the other half*** $|\downarrow\rangle_z$.

Design an experiment to differentiate these two beams of atoms. (You should be able to tell after your experiment that one of the beams is in a pure state and the other is a mixture.)

14. Read the following statement and answer the questions.

Andy: There is no difference between silver atoms in a "pure" state given by $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$ and an unpolarized mixture in which half of the atoms are in the $|\uparrow\rangle_z$ state and half are in the $|\downarrow\rangle_z$ state. If we had sent atoms in the superposition state $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_z + |\downarrow\rangle_z\right)$ through the SGZ, half of them would have registered in the "up" detector and half of them would have been collected in the lower channel. The outcome will be exactly the same if we had sent a 50/50 mixture of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ through the SGZ. So there is no way to distinguish a mixture from a superposition.

Question : Is the statement above correct? Explain.



15. Remember the analogy between spin states and polarized photons in *question 13*. Suppose you have a beam of pure polarized photons with 45 degrees polarization and another beam of unpolarized mixture with half of the photons vertically polarized and half horizontally polarized. Will a vertical or horizontal polarizer tell you which beam is in a pure state? What polarizer could you use to differentiate the two beams of photons?

16. Based upon the analogy for distinguishing between pure polarized photons and a beam of photon mixture, what kind of SGA could you use to differentiate the two beams of atoms in *question 14*? Draw a sketch below to explain your choice. Do not forget to put the detectors in the correct positions. (Hint: Use the simulations "z-up pass SGX" and "z-up+z-down pass SGX" to check your answer. In the simulation "z-up+z-down pass SGX", the incoming particles are in the pure state $\frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_z + \left|\downarrow\right\rangle_z\right)$.)

Questions 17 and 18 relate to the simulation "unknown state". Run the simulation "unknown state" first. Then answer the following questions.

17. Write down at least 3 different possible spin states of the incoming particles that will show the behavior seen in the simulation. The incoming particles do not necessarily have identical spin states. Explain your reasoning for your choices.

18. Choose two of the different possible spin states you predicted for the simulation you saw. Now come up with some simulations using SGAs that would distinguish between the two possible spin states. You can choose one or more SGAs to find out which of the two spin states it is. Share your set-up with others in your class.

Instructions on building your own SGA:

(1) Open the simulation "unknown state" or any other existing simulation.

(2) Choose "File→New" on the simulation menus. Then you would have a white board.

(3) Click the first button "New Gun" to add a particle source to the white board. You can click the icon on the white board and drag it to a new position.

(4) Choose "Initialize→User State" to set up the initial spin state of the particle source. You can choose the basis X, Y or Z regarding spin Sx, Sy and Sz. And then you can input the coefficient of spin-up and spin-down in the table. Take Sx as an example.

System	Real 1	Imag 1	Real 2	Imag 2
Spin 1/2	*1*	*2*	*3*	*4*

The coefficient in the table above means the initial state is $(1+2i)\left|\uparrow_x\right\rangle + (3+4i)\left|\downarrow_x\right\rangle$

*Remember to press "Enter" on the keyboard after you have input all the numbers.

5. Click the second button "New Analyzer" to add the Stern-Gerlach apparatus. The letter "X", "Y" and "Z" represent the direction of the magnetic field gradient. You can change the letter by clicking it. The letter "n" represent a customized direction which could be defined by "Design→Change Angles".

6. Click the fourth button "New Counter" to add the detector. Drag it to the proper position.

7. Connect the particle source and the SGA by clicking the end on the right side of the particle source and dragging a line to the left side of the SGA. Then click the upper or lower channel of the SGA and drag a line to the detector to connect them together.

8. Click "Go" to test whether the simulation works well.