An electron in a uniform magnetic field: Larmor precession of spin

For reference, if we choose the eigenstates of $\hat{S}_z$ as the basis vectors, the components of the spin angular momentum are given by:
$S_z = \hbar/2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $S_x = \hbar/2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $S_y = \hbar/2 \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$.
The following identities may be useful:
$(e^{i\theta} + e^{-i\theta})/2 = \cos(\theta)$, $(e^{i\theta} - e^{-i\theta})/(2i) = \sin(\theta)$
$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$, $2\sin(\theta)\cos(\theta) = \sin(2\theta)$, $\cos^2(\theta) + \sin^2(\theta) = 1$

1. In quantum mechanics, the energy associated with the spin magnetic moment has a corresponding Hamiltonian operator $\hat{H}$ for the spin degrees of freedom. If the uniform magnetic field is $\vec{B} = B_0\hat{k}$, then $\hat{H} = -\gamma B_0 \hat{S}_z$. Which one of the following is the $\hat{H}$ in the matrix form in the chosen basis?
(a) $-\gamma B_0 \hbar/2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
(b) $-\gamma B_0 \hbar/2 \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$
(c) $-\gamma B_0 \hbar/2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
(d) $-\gamma B_0 \hbar/2 \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$

2. If we measured the energy of the above system, which one of the following gives the allowed values of energy E_+ and E_-?
(a) $\mp\gamma B_0 \hbar/2$
(b) $\mp\hbar/2$
(c) $\mp\gamma B_0 \hbar$
(d) None of the above

3. Consider the following conversation between Andy and Caroline about the above Hamiltonian operator:
• Andy: $\hat{H}$ is essentially $\hat{S}_z$ except for some multiplicative constants. Therefore, the eigenstates of $\hat{S}_z$ will also be the eigenstates of $\hat{H}$.
• Caroline: No. The presence of magnetic field will make the eigenstates of $\hat{S}_z$ and $\hat{H}$ different. The eigenstates of $\hat{H}$ will change with time in a non-trivial manner.
• Andy: I disagree. If the magnetic field had a time dependence, e.g., $B = B_0 \cos(\omega t)\hat{k}$, the eigenstates of $\hat{H}$ will change with time in a non-trivial manner but not for the present case where $\vec{B}$ is constant.
With whom do you agree?
(a) Andy
(b) Caroline
(c) Neither

4. If the eigenstates of $\hat{S}_z$ and $\hat{H}$, $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$, are chosen as the basis vectors, which one of the following is their matrix representation?
   (a) $\binom{1}{0}$ and $\binom{0}{1}$
   (b) $\binom{1}{1}$ and $\binom{0}{0}$
   (c) $1/\sqrt{2}\binom{1}{1}$ and $1/\sqrt{2}\binom{1}{-1}$
   (d) $1/\sqrt{2}\binom{1}{i}$ and $1/\sqrt{2}\binom{1}{-i}$

5. If we choose $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ as the basis vectors for the two dimensional spin space, which one of the following is the correct expression for a general state $|\chi\rangle$?
   (a) $|\chi\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$ where $|a| + |b| = 1$.
   (b) $|\chi\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$ where $|a|^2 + |b|^2 = 1$.
   (c) $|\chi\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$ where a and b can be any integers.
   (d) $|\chi\rangle = a|\uparrow\rangle_z \times b|\downarrow\rangle_z$ where a and b can be any integers.

6. If the state of the system at an initial time $t = 0$ is given by $|\chi(t = 0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, which one of the following is the correct matrix representation of this state in the chosen basis?
   (a) $\binom{a}{b}$
   (b) $\binom{a-b}{a+b}$
   (c) $\binom{a+b}{a-b}$
   (d) $1/\sqrt{2}\binom{a}{b}$

7. If the state of the system at an initial time $t = 0$ is given by $|\chi(t = 0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, which one of the following is the state, $|\chi(t)\rangle$, after a time t?
   (a) $e^{i\gamma B_0 t/2}(a|\uparrow\rangle_z + b|\downarrow\rangle_z)$
   (b) $e^{-i\gamma B_0 t/2}(a|\uparrow\rangle_z + b|\downarrow\rangle_z)$
   (b) $e^{i\gamma B_0 t/2}[(a+b)|\uparrow\rangle_z + (a-b)|\downarrow\rangle_z]$
   (d) $e^{i\gamma B_0 t/2}a|\uparrow\rangle_z + e^{-i\gamma B_0 t/2}b|\downarrow\rangle_z$

8. If the state of the system at an initial time $t = 0$ is given by $|\chi(t = 0)\rangle = a|\uparrow\rangle_z + b|\downarrow\rangle_z$, which one of the following is the correct matrix representation of this state, $|\chi(t)\rangle$, after a time t?
   (a) $e^{i\gamma B_0 t/2}\binom{a}{b}$
   (b) $e^{-i\gamma B_0 t/2}\binom{a}{b}$
   (c) $e^{-i\gamma B_0 t/2}\binom{a+b}{a-b}$
   (d) $\begin{pmatrix} ae^{i\gamma B_0 t/2} \\ be^{-i\gamma B_0 t/2} \end{pmatrix}$

Note: All of the following questions refer to the system for which the Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

In all of the questions below, $|\chi(t)\rangle$ is the state you calculated above and assume that the coefficients a and b in the state are non-zero unless mentioned specifically otherwise.

9. Which one of the following gives the correct outcomes of measuring S_z in the state $|\chi(t)\rangle$?

(a) $\hbar/2$ with a probability $ae^{i\gamma B_0 t/2}$ and $-\hbar/2$ with a probability $be^{-i\gamma B_0 t/2}$.

(b) $\hbar/2$ with a probability $a^2 e^{i\gamma B_0 t}$ and $-\hbar/2$ with a probability $b^2 e^{-i\gamma B_0 t}$.

(c) $\hbar/2$ with a probability $|a|^2$ and $-\hbar/2$ with a probability $|b|^2$.

(d) $\hbar/2$ and $-\hbar/2$ with equal probability.

10. Consider the following conversation between Andy and Caroline about measuring $\hat{S}_z$ in the state $|\chi(t)\rangle$:

• Andy: Since the probability of measuring $\hbar/2$ is $|a|^2$, $-\hbar/2$ is $|b|^2$ and $|a|^2 + |b|^2 = 1$, we can choose our a and b as $a = e^{i\phi_1}\cos(\alpha)$ and $b = e^{i\phi_2}\sin(\alpha)$ where α, ϕ_1 and ϕ_2 are real numbers.

• Caroline: I agree. Since $\cos^2(\alpha) + \sin^2(\alpha) = 1$ it gives the same relation as $|a|^2 + |b|^2 = 1$ and there is no loss of generality.

Do you agree with Andy and Caroline?

(a) Yes.

(b) No.

11. Calculate the expectation value of $\hat{S}_z$ in the state $|\chi(t)\rangle$. Express your answer in terms of α defined earlier as $a = e^{i\phi_1}\cos(\alpha)$ and $b = e^{i\phi_2}\sin(\alpha)$.

12. Consider the following conversation between Pria and Mira about $\langle \hat{S}_z \rangle$ in state $|\chi(t)\rangle$:

• Pria: Since the state of the system $|\chi(t)\rangle$ evolves in time, the expectation value $\langle S_z \rangle$ will depend on time.

• Mira: I disagree with the second part of your statement. The time development of the expectation value of any operator $\hat{A}$ is given by $\frac{d\langle \hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle + \langle\frac{\partial \hat{A}}{\partial t}\rangle$. In our case, $[\hat{H}, \hat{S}_z] = 0$ and the operator $\hat{S}_z$ does not have any explicit time dependence so $\frac{\partial \hat{S}_z}{\partial t} = 0$. Thus, $\frac{d\langle \hat{S}_z\rangle}{dt} = 0$ and the expectation value will not change with time.

With whom do you agree?

(a) Pria

(b) Mira

(c) Neither

13. Which one of the following is the expectation value $\langle \hat{S}_z \rangle = \langle \chi(t)|\hat{S}_z|\chi(t)\rangle$?
   (a) $\cos(2\alpha)\ \cos(\gamma B_0 t)\ \hbar/2$
   (b) $\sin(2\alpha)\ \sin(\gamma B_0 t)\ \hbar/2$
   (c) $\cos(2\alpha)\ \hbar/2$
   (d) $\sin(2\alpha)\ \hbar/2$

14. Which one of the following is true about the expectation value $\langle \hat{S}_x \rangle$ in the state $|\chi(t)\rangle$?
   (a) $\langle \hat{S}_x \rangle$ depends on time because $[\hat{H}, \hat{S}_x] \neq 0$
   (b) $\langle \hat{S}_x \rangle$ depends on time because $\hat{S}_x$ has an explicit time-dependence.
   (c) $\langle \hat{S}_x \rangle$ is time-independent because $[\hat{H}, \hat{S}_x] = 0$
   (d) None of the above.

15. Calculate the expectation value of $\hat{S}_x$ in the state $|\chi(t)\rangle$. Express your answer in terms of α defined earlier as $a = e^{i\phi_1}\cos(\alpha)$ and $b = e^{i\phi_2}\sin(\alpha)$.

16. Which one of the following is the expectation value $\langle \hat{S}_x \rangle$ in the state $|\chi(t)\rangle$?
   (a) $\cos(2\alpha)\ \cos(\gamma B_0 t + \phi_1 - \phi_2)\ \hbar/2$
   (b) $\sin(2\alpha)\ \cos(\gamma B_0 t + \phi_1 - \phi_2)\ \hbar/2$
   (c) $\cos(2\alpha)\ \hbar/2$
   (d) $\sin(2\alpha)\ \hbar/2$

17. Choose all of the following statements that are true about the expectation value $\langle \hat{S}_y \rangle$ in the state $|\chi(t)\rangle$:

   (I) $\langle \hat{S}_y \rangle$ depends on time because $[\hat{H}, \hat{S}_y] \neq 0$
   (II) $\langle \hat{S}_y \rangle$ depends on time because $\hat{S}_y$ has an explicit time-dependence.
   (III) $\langle \hat{S}_y \rangle$ is time-independent because $[\hat{H}, \hat{S}_y] = 0$

   (a) (I) only
   (b) (II) only
   (c) (III) only
   (d) (I) and (II) only

18. Based upon your responses for $\langle \hat{S}_z \rangle$ and $\langle \hat{S}_x \rangle$ above, which one of the following is a good guess for the expectation value $\langle \hat{S}_y \rangle$ in state $|\chi(t)\rangle$? Note: Please work this out explicitly as homework similar to your calculation for $\langle \hat{S}_x \rangle$ above.

    (a) $\cos(2\alpha)\,\cos(\gamma B_0 t + \phi_1 - \phi_2)\,\hbar/2$

    (b) $-\sin(2\alpha)\,\sin(\gamma B_0 t + \phi_1 - \phi_2)\,\hbar/2$

    (c) $\cos(2\alpha)\,\hbar/2$

    (d) $\sin(2\alpha)\,\hbar/2$

19. Explain why $\langle \hat{S}_z \rangle$ above does not depend on time where as $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ do.

20. Choose all of the following statements that are true about the expectation value $\langle \vec{S} \rangle$ in the state $|\chi(t)\rangle$:

    (I) $\langle \hat{S} \rangle = \langle \hat{S}_x \rangle \hat{\imath} + \langle \hat{S}_y \rangle \hat{\jmath} + \langle \hat{S}_z \rangle \hat{k}$

    (II) $\langle \hat{S} \rangle$ will depend on time because $[\hat{H}, \hat{S}] \neq 0$

    (III) $\langle \hat{S} \rangle$ cannot depend on time because the expectation value of an observable is its time-averaged value.

    (a) (I) only

    (b) (II) only

    (c) (III) only

    (d) (I) and (II) only

21. Choose all of the following statements that are true about the expectation value $\langle \hat{S} \rangle$ in the state $|\chi(t)\rangle$:

    (I) The z component of $\langle \hat{S} \rangle$, i.e., $\langle \hat{S}_z \rangle$, is time-independent.

    (II) The x and y components of $\langle \hat{S} \rangle$ change with time and they are always "out of phase" with each other for all times.

    (III) The magnitude of the maximum values of $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ are the same but when $\langle \hat{S}_x \rangle$ is a maximum $\langle \hat{S}_y \rangle$ is a minimum and vice versa.

    (a) (I) and (II) only

    (b) (I) and (III) only

    (c) (II) and (III) only

    (d) (I), (II) and (III)

22. Choose all of the following statements that are true about the vector $\langle \hat{S} \rangle$ in the spin space in the state $|\chi(t)\rangle$:

(I) The vector $\langle \hat{S} \rangle$ can be thought to be precessing about the z axis at an angle 2α.

(II) The vector $\langle \hat{S} \rangle$ can be thought to be precessing about the z axis with a frequency $\omega = \gamma B_0$.

(III) All the three components of vector $\langle \hat{S} \rangle$ change as it precesses about the z axis.

(a) (I) and (II) only

(b) (I) and (III) only

(c) (II) and (III) only

(d) (I), (II) and (III)

23. Choose a three dimensional coordinate system in the spin space with the z axis in the vertical direction. Draw a sketch showing the precession of $\langle \hat{S} \rangle$ about the z axis when the state of the system starts out in $a|\uparrow\rangle_z + b|\downarrow\rangle_z$. Show the angle that $\langle \hat{S} \rangle$ makes with the z axis and the precession frequency explicitly.

Show the projection of $\langle \hat{S} \rangle$ along the x, y and z axes at two separate times. Explain in words why the projection of $\langle \hat{S} \rangle$ along the z direction does not change with time but those along the x and y directions change with time.

Now consider a very specific initial state which is an eigenstate of $\hat{S}_z$, e.g., $|\uparrow\rangle_z$, in the following questions (as opposed to a very general initial state $a|\uparrow\rangle_z + b|\downarrow\rangle_z$ in the previous questions).

24. If the electron is initially in the state $|\chi(0)\rangle = |\uparrow\rangle_z$, write the state of the system $|\chi(t)\rangle$ after a time t. The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

25. Evaluate the expectation values of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ at time t in the above state. Which of these expectation values depend on time?

26. Explain why each expectation value you calculated in the previous question does or does not depend on time.

27. Calculate the expectation value $\langle\chi(t)|[\hat{H}, \hat{S}_x]|\chi(t)\rangle$ in the above state by writing $\hat{H} = -\gamma B_0 \hat{S}_z$ explicitly and acting with $\hat{S}_z$ on the state $|\chi(t)\rangle$.

28. Since $\frac{d\langle\hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle + \langle\frac{\partial\hat{A}}{\partial t}\rangle$ what can you infer about the time-dependence of $\langle\hat{S}_x\rangle$ in the state $|\uparrow\rangle_z$ from your last response? What about $\langle\hat{S}_y\rangle$ or $\langle\hat{A}\rangle$ where the operator $\hat{A}$ does not have an explicit time-dependence?

29. Consider the following statements from Pria and Mira when the electron is initially in an eigenstate of $\hat{S}_z$. The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

   • Pria: The electron will NOT be in an eigenstate of $\hat{S}_z$ forever because the state will evolve in time.

   • Mira: I disagree. The eigenstates of $\hat{S}_z$ are also the eigenstates of $\hat{H}$. When the system is in an energy eigenstate or a stationary state, the time dependence is via an overall phase factor. The system stays in the stationary state. Since the system is in a stationary state, the expectation value of ANY operator (that does not have an explicit time dependence) will not depend on time as we saw due to $\langle\chi(t)|[\hat{H}, \hat{S}_x]|\chi(t)\rangle = 0$.

   With whom do you agree? Explain why the other person is not correct.

   (a) Pria

   (b) Mira

30. Consider the following statements from Pria and Mira when the electron is initially in an eigenstate of $\hat{S}_z$. The Hamiltonian operator is $\hat{H} = -\gamma B_0 \hat{S}_z$.

- Pria: But shouldn't the expectation value $\langle \vec{S} \rangle = \langle S_x \rangle \hat{i} + \langle S_y \rangle \hat{j} + \langle S_z \rangle \hat{k}$ precess about the z axis whether the state is $|\uparrow\rangle_z$ or $a|\uparrow\rangle_z + b|\downarrow\rangle_z$?
- Mira: No. Since $|\uparrow\rangle_z$ is an eigenstate of the Hamiltonian, it is a stationary state which has a trivial time dependence. $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ do not depend on time since the state will remain an energy eigenstate and will evolve by a trivial overall time-dependent phase factor as $e^{-iE_+t/\hbar}|\uparrow\rangle_z$. There is no precession.
- Pria: How can that be?
- Mira: Another way to reason about it is by comparing $|\uparrow\rangle_z$ with $a|\uparrow\rangle_z + b|\downarrow\rangle_z$. If our state is $|\uparrow\rangle_z$ then $a = e^{i\phi_1}\cos(\alpha) = 1$ and $b = e^{i\phi_2}\sin(\alpha) = 0$ which implies that $\alpha = 0$ and $\langle \vec{S} \rangle$ is pointing along the z direction. Thus, there is no precession. If you calculate $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ in this state, they will both be always zero since the projection of $\langle \vec{S} \rangle$ along the x and y axis is zero.

With whom do you agree? Explain.

(a) Pria

(b) Mira

31. Calculate $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ in the state $e^{-iE_+t/\hbar}|\uparrow\rangle_z$ to verify your response above.

32. If the electron is initially in the state $|\downarrow\rangle_z$, what is $|\chi(t)\rangle$ at a time t. Find the angle that $\langle \vec{S} \rangle$ makes with the z axis. Is there any precession in this case? Explain.

33. Choose all of the following statements that are true if the system is initially in an eigenstate of $\hat{S}_z$, $|\uparrow\rangle_z$, and the Hamiltonian operator $\hat{H} = -\gamma B_0 \hat{S}_z$:

(I) $\langle \hat{S}_x \rangle$ depends on time because $[\hat{H}, \hat{S}_x] \neq 0$

(II) $\langle \hat{S}_y \rangle$ depends on time because $[\hat{H}, \hat{S}_y] \neq 0$

(III) The vector $\langle \hat{S} \rangle$ can be thought to be precessing about the z axis.

(a) (I) and (II) only

(b) (III) only

(c) (I), (II) and (III)

(d) None of the above

34. If $\hat{H} = -\gamma B_0 \hat{S}_z$ and the electron is initially in the state $|\uparrow\rangle_x$, what is the state after time t?

35. Based upon your answer to the preceding question, should the expectation value $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$ and $\langle \hat{S}_z \rangle$ depend on time if the initial state is an eigenstate of $\hat{S}_x$, i.e., $|\uparrow\rangle_x$? Work out each of these expectation values at time t to justify your answer for each case.

36. Consider the following conversation between three students about the preceding problem in which the Hamiltonian for the system $\hat{H} = -\gamma B_0 \hat{S}_z$:
Student A: If the electron is initially in an eigenstate of $\hat{S}_x$, expectation value of $\hat{S}_x$ is time-independent because the electron is stuck in the eigenstate.
Student B: I disagree. If the electron is initially in an eigenstate of $\hat{S}_x$, only the expectation value of $\hat{S}_z$ is time independent because $\hat{S}_z$ and the Hamiltonian H commute with each other. Since an eigenstate of $\hat{S}_x$ is not a stationary state, the time-dependence of the eigenstate of $\hat{S}_x$ is non-trivial. You can see this by writing the eigenstate of $\hat{S}_x$, e.g., $|\uparrow\rangle_x$, in terms of eigenstates of the Hamiltonian at time $t = 0$ and then writing down the state after time t explicitly.
Student C: S_z is a constant of motion since $\hat{S}_z$ and the Hamiltonian H commute with each other. Therefore, $\langle \hat{S}_z \rangle$ is time-independent in any state. With whom do you agree? Explain.