Name ________________________________

Identical Particles

1 Notes for this tutorial:

- We will only consider systems of non-interacting identical particles.
- The word "identical" in this tutorial will refer to one type of particle (all particles with the same properties). For example, all electrons are identical.
- Assume that all systems with more than one particle consist of identical particles. For example, a system of fermions is made up of identical fermions (e.g., electrons) and a system of bosons is made up of identical bosons (e.g., Helium-4 atoms).
- Identical particles (particles of one type with the same properties) are in general indistinguishable (e.g., you cannot distinguish which particle is in which single particle stationary state). Exchanging these indistinguishable particles with each other does not produce a distinctly different many-particle state.
- Assume that particles are restricted to one spatial dimension (spatial coordinate given by x) for convenience.
- We will use the notation $\hat{H}_i$ to denote the Hamiltonian in the M-dimensional Hilbert space for the i^{th} particle. We will use the boldface notation $\hat{\mathbf{H}}_i$ to denote the Hamiltonian of the i^{th} particle in the M^N-dimensional Hilbert space for the N-particle system.
- Unless otherwise stated, the single-particle wavefunction, $\psi_n(x)$, in this tutorial refers to the normalized single-particle stationary state wavefunction.
- The N-particle wavefunction, $\psi(x_1, x_2, \cdots, x_N) = \psi_{n_1,n_2,\cdots,n_N}(x_1, x_2, \cdots, x_N)$, in this tutorial refers to the many-particle stationary state wavefunction with coordinates $x_1, x_2, \ldots, x_N$ for different particles.
- The wavefunction of a system of two non-interacting identical particles has terms such as $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, where $\psi_{n_1}(x_1)$ and $\psi_{n_2}(x_2)$ are the single-particle wavefunctions for particles in states n_1 and n_2 and coordinates x_1 and x_2, respectively.
  - Remark: $\psi_{n_1}(x_1)$ and $\psi_{n_2}(x_2)$ should be regarded as any single-particle wavefunctions for particles 1 and 2, respectively (i.e., in general, ψ_{n_1} does not refer to the ground state and ψ_{n_2} does not refer to the first-excited state wavefunction).
- Here, for convenience, we will refer to all direct products of single-particle states as "basis states". Please note that for identical fermions, only completely antisymmetric linear combinations of these basis states are allowed, while for bosons only completely symmetric linear combinations are allowed. For distinguishable particles, all basis states are allowed.
- The energy of the system of N non-interacting identical particles is given by $E = E_{n_1} + E_{n_2} + \cdots + E_{n_N} = \sum_{i=1}^{N} E_{n_i}$, in which E_{n_i} is the energy corresponding to the single-particle state ψ_{n_i}.
- Unless otherwise specified, there is no degeneracy in the energy spectrum of the single-particle states. That is $E_{n_i} \neq E_{n_j}$ for $n_i \neq n_j$, in which E_{n_i} is the energy corresponding to the single-particle state with wavefunction ψ_{n_i} and E_{n_j} is the energy corresponding to the single-particle state ψ_{n_j}.
- Unless otherwise specified, assume that the particles are spinless for the purposes of constructing the many-particle wavefunction and ignore the spin part of the wavefunction.

- The product notation, e.g, $\prod_{i=1}^{N} x_i$, will be used to represent the product of x_i for $i = 1, 2, \dots, N$ (i.e. $\prod_{i=1}^{N} x_i = x_1 x_2 x_3 \cdots x_N$).

2 Objectives

Upon completion of this tutorial, you should be able to do the following:

1. Determine the form of the Hamiltonian for non-interacting identical particles.
2. Determine the basis states in the product space for a system of non-interacting identical particles
3. Determine the form of the wavefunction for a system of non-interacting identical particles if the particles are indistinguishable fermions, indistinguishable bosons, or a hypothetical case in which identical particles can be treated as distinguishable.
4. Construct the wavefunction for the ground state and first-excited state for a specific two-particle system for two non-interacting identical particles (particles of one type with the same properties) if the particles are:
   (a) Indistinguishable bosons
   (b) Indistinguishable fermions
   (c) Hypothetical case: Identical particles which can be treated as distinguishable
5. Determining the Number of Distinct Many-Particle States
   (a) CASE 1: The total energy of the many-particle system is not fixed, but a fixed number of single-particle states are available to the system:
      i. Calculate the number of distinct many-particle states if you have two particles, three particles, or N particles ($N \gg 1$) in the following cases:
         A. Particles are indistinguishable bosons
         B. Particles are indistinguishable fermions
         C. Hypothetical case: Identical particles which can be treated as distinguishable
      ii. Compare the results for the cases of indistinguishable bosons and indistinguishable fermions to the results for the hypothetical case when identical particles can be treated as distinguishable.
   (b) CASE II: The total energy of the many-particle system is fixed:
      i. Calculate the number of distinct many-particle states if you have two particles or three particles in the following cases:
         A. Particles are indistinguishable bosons
         B. Particles are indistinguishable fermions
         C. Hypothetical case: Identical particles which can be treated as distinguishable
      ii. Compare the results for the cases of indistinguishable bosons and indistinguishable fermions to the results for the hypothetical case when identical particles can be treated as distinguishable.
      iii. For a system of two non-interacting identical particles, determine the probability of obtaining a particular value of the energy of a particle when the single-particle energy is measured at random and the total energy is fixed for a specified many-particle system if the particles are:
         A. Indistinguishable bosons
         B. Indistinguishable fermions

C. Hypothetical case: Identical particles which can be treated as distinguishable

iv. Compare the results for the cases of indistinguishable bosons and indistinguishable fermions to the results for the hypothetical case when identical particles can be treated as distinguishable.

(c) CASE III: The single-particle states have degeneracy and the total energy of the many-particle system is fixed by fixing the number of particles in each group of degenerate single-particle states with a given energy.

i. Calculate the number of distinct many-particle states in the following cases:

A. Particles are indistinguishable bosons

B. Particles are indistinguishable fermions

C. Hypothetical case: Identical particles which can be treated as distinguishable.

6. Determine the wavefunction including spin for a system of non-interacting identical particles if the particles are indistinguishable fermions or bosons.

7. Construct the wavefunction for the ground state and first-excited state for specific many-particle system for many non-interacting identical particles if the particles are:

   (a) Indistinguishable bosons

   (b) Indistinguishable fermions.

8. Determine the form of the wavefunction for a system of non-interacting identical particles in the limiting case when identical particles can be treated as distinguishable.

3 Basics for a System of N Non-Interacting Particles

3.1 Hamiltonian for a System of Non-interacting Particles

- Before we determine the form of the stationary state wavefunction for a system of N non-interacting identical particles, let's determine the form of the Hamiltonian for a system of non-interacting particles in terms of the single-particle Hamiltonian.
- We will use the notation $\hat{H}_i$ to denote the Hamiltonian in the M-dimensional Hilbert space for the i^{th} particle. We will use the boldface notation $\hat{\mathbf{H}}_i$ to denote the Hamiltonian of the i^{th} particle in the M^N-dimensional Hilbert space for the many-particle system.
- The following question and conversations will guide you as you think about the Hamiltonian for a system of N non-interacting identical particles in which each particle is in a M-dimensional space.

1. Write the Hamiltonian $\hat{\mathbf{H}}$ for a system of N non-interacting, identical particles in the product space in terms of the Hamiltonians for the individual particles $\hat{\mathbf{H}}_i$ $(i = 1, 2, \ldots, N)$..

Consider the following conversation regarding constructing the Hamiltonian for a system of N **non-interacting** identical particles in which each particle is in a M-dimensional space.

Student 1: The Hamiltonian for the non-interacting N-particle system in the M^N-dimensional product space is $\hat{\mathbf{H}} = \hat{\mathbf{H}}_1 \otimes \hat{\mathbf{H}}_2 \otimes \hat{\mathbf{H}}_3 \otimes \cdots \otimes \hat{\mathbf{H}}_N$, in which $\hat{\mathbf{H}}_i = \hat{I}_1 \otimes \hat{I}_2 \otimes \cdots \otimes \hat{I}_{i-1} \otimes \hat{H}_i \otimes \hat{I}_{i+1} \cdots \otimes \hat{I}_N$ is the Hamiltonian of the i^{th} particle in the M^N-dimensional space. The single-particle Hamiltonian, $\hat{H}_i$, and the identity operator, $\hat{I}_i$, are for the i^{th} particle in the M-dimensional space.

Student 2: I disagree with Student 1. The Hamiltonian $\hat{\mathbf{H}}$ for non-interacting particles in the M^N-dimensional product space is $\hat{\mathbf{H}} = \hat{H}_1 \otimes \hat{H}_2 \otimes \hat{H}_3 \otimes \cdots \otimes \hat{H}_N$.

Student 3: I disagree with Student 1 and Student 2. If we know the single-particle Hamiltonian $\hat{H}_i$ for the i^{th} particle in the system in the M-dimensional space, then the Hamiltonian for a system of N non-interacting identical particles in the M^N-dimensional product space has the form $\hat{\mathbf{H}} = \hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_N$.

Student 4: I disagree with Student 1, Student 2, and Student 3. Since the Hamiltonian for the system must be in the M^N-dimensional product space, $\hat{\mathbf{H}} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2 + \cdots + \hat{\mathbf{H}}_N$. The single-particle Hamiltonian for the i^{th} particle in the M^N-dimensional product space is $\hat{\mathbf{H}}_i = \hat{I}_1 \otimes \hat{I}_2 \otimes \cdots \hat{I}_{i-1} \otimes \hat{H}_i \otimes \hat{I}_{i+1} \otimes \cdots \otimes \hat{I}_N$, where the boldface notation $\hat{\mathbf{H}}_i$ is for the M^N-dimensional product space. The sum of the M-dimensional single-particle Hamiltonians $\hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_N$ is only M-dimensional and is not in the M^N-dimensional product space.

Explain why you agree or disagree with each student.

Consider the following conversation regarding constructing the Hamiltonian for a system of N **non-interacting** identical particles in which each particle is in a M-dimensional space.

Student 1: If we know the single-particle Hamiltonian $\hat{\mathbf{H}}_i$ for the i^{th} particle in the system in the M^N-dimensional space, then the Hamiltonian for a system of N non-interacting identical particles has the form $\hat{\mathbf{H}} = (\hat{H}_1 \otimes \hat{I}_2 \otimes \hat{I}_3 \otimes \cdots \otimes \hat{I}_N) + (\hat{I}_1 \otimes \hat{H}_2 \otimes \hat{I}_3 \otimes \cdots \otimes \hat{I}_N) + \cdots + (\hat{I}_1 \otimes \hat{I}_2 \otimes \cdots \otimes \hat{I}_{N-2} \otimes \hat{H}_{N-1} \otimes \hat{I}_N) + (\hat{I}_1 \otimes \hat{I}_2 \otimes \cdots \otimes \hat{I}_{N-1} \otimes \hat{H}_N)$, with the single-particle Hamiltonian, $\hat{H}_i$, and the identity operator, $\hat{I}_i$, for the i^{th} particle in the M-dimensional space.

Student 2: I agree with Student 1. Since the particles are non-interacting, the Hamiltonian $\hat{\mathbf{H}}_i$ for the i^{th} particle is not entangled with the Hamiltonian $\hat{\mathbf{H}}_j$ for the j^{th} particle. A short hand notation for the sum is $\hat{\mathbf{H}} = \sum_{i=1}^{N} \hat{\mathbf{H}}_i = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2 + \hat{\mathbf{H}}_3 + \cdots + \hat{\mathbf{H}}_N$.

Explain why you agree or disagree with each student.

** CHECKPOINT: Check your answer to question 1. **

1. $\hat{\mathbf{H}} = \sum_{i=1}^{N} \hat{\mathbf{H}}_i = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2 + \hat{\mathbf{H}}_3 + \cdots + \hat{\mathbf{H}}_N$

If your answer does not match the checkpoint, go back and reconcile any difference you may have with the checkpoint answer.

Consider the following conversation regarding two non-interacting identical particles in a one-dimensional infinite square well.

Student 1: In an infinite square well, we are only permitted to have one-particle in the well. If the system has two non-interacting identical particles, we MUST have two infinite square wells in order to place each particle.

Student 2: I disagree. We can have two non-interacting identical particles in the same infinite square well. If the particles are non-interacting and confined to a well of width a, the Hamiltonian for each particle in the product space will be $\hat{\mathbf{H}}_i = \frac{\hat{p}_i^2}{2m} + V(x_i)$, in which

$$V(x_i) = \begin{cases} 0 & \text{if} \quad 0 \leq x_i \leq a \\ \infty & \text{otherwise} \end{cases} \qquad (i = 1, 2).$$

The Hamiltonian for the system of two non-interacting identical particles in the same well in the product space is $\hat{\mathbf{H}} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2 = \hat{H}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{H}_2$, where $\hat{\mathbf{H}}_1$ and $\hat{\mathbf{H}}_2$ are the single-particle Hamiltonians in the product space and $\hat{H}_1$ and $\hat{H}_2$ are the single-particle Hamiltonians in the subspaces for the individual particles.

Explain why you agree or disagree with each student.

Summary of the Hamiltonian for a System of N Non-interacting Particles.

- The Hamiltonian $\hat{\mathbf{H}}$ for a system of N non-interacting particles in the product space is the sum of the Hamiltonians for each particle in the product space, $\hat{\mathbf{H}} = \sum_{i=1}^{N} \hat{\mathbf{H}}_i = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2 + \hat{\mathbf{H}}_3 + \cdots + \hat{\mathbf{H}}_N$ with $\hat{\mathbf{H}}_i = \hat{I}_1 \otimes \hat{I}_2 \otimes \cdots \otimes \hat{I}_{i-1} \otimes \hat{H}_i \otimes \hat{I}_{i+1} \otimes \cdots \otimes \hat{I}_N$.

3.2 Determining Whether the Basis States in the Product Space for a System of N Non-Interacting Identical Particles Should be Written in Terms of the Sum or Product of the Single-Particle Stationary State Wavefunctions

- Now that we know the form of the Hamiltonian $\hat{\mathbf{H}}$ for a system of N non-interacting identical particles in terms of the single-particle Hamiltonian $\hat{\mathbf{H}}_i$ in the product space, let's think about the form of the stationary state wavefunction for this system.
- The form of the stationary state wavefunction for a system of non-interacting identical particles will depend on the type of particle. We will consider three cases:
  - Indistinguishable fermions
  - Indistinguishable bosons
  - Hypothetical case: Identical particles which can be treated as distinguishable.
- Here, for convenience, we will refer to all direct products of single-particle states as "basis states". Please note that for identical fermions, only completely antisymmetric linear combinations of these basis states are allowed, while for bosons only completely symmetric linear combinations are allowed. For distinguishable particles, all basis states are allowed.
  - Let's consider the appropriate basis states, e.g., whether the wavefunction for a system of N non-interacting identical particles can be written in terms of the sum or the product of the single-particle wavefunctions of individual particles.

2. Explain why you agree or disagree with the following student. If you disagree, write a correct statement.

   Student 1: The wavefunction $\psi_{n_1}(x_1)$ describes a particle in a single-particle state denoted by quantum number n_1 specifying a single-particle energy and coordinate x_1.

3. Write the right-hand side without operators, if possible, in the following questions for a system of two non-interacting identical particles, whose single-particle wavefunctions satisfy the Time Independent Schrödinger Equation (TISE), $\hat{H}_i\psi_{n_j}(x_i) = E_{n_j}\psi_{n_j}(x_i)$ for the i^{th} particle with coordinate x_i in the single-particle state given by n_j. Assume $n_1 \neq n_2$. If it is not possible to write the right-hand side without operators and without encountering difficulties or inconsistencies, explain why.

   (a) $\hat{\mathbf{H}}_1[\psi_{n_1}(x_1) + \psi_{n_2}(x_2)] =$ ________________________

   (b) $\hat{\mathbf{H}}_2[\psi_{n_1}(x_1) + \psi_{n_2}(x_2)] =$ ________________________

   (c) $(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)[\psi_{n_1}(x_1) + \psi_{n_2}(x_2)] =$ ________________________

   (d) $\hat{\mathbf{H}}_1[\psi_{n_1}(x_1)\psi_{n_2}(x_2)] =$ ________________________

   (e) $\hat{\mathbf{H}}_2[\psi_{n_1}(x_1)\psi_{n_2}(x_2)] =$ ________________________

   (f) $(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) =$ ________________________

   (g) $(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)] =$ ____________________________________

   (h) $(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)] =$ ____________________________________

4. Circle all of the following wavefunctions "Ψ" (taken from question 3) that are "possible" two-particle stationary state wavefunctions. Ignore normalization. (Hint: The wavefunction Ψ should satisfy $\hat{\mathbf{H}}\Psi = E\Psi$ in which $\hat{\mathbf{H}} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$ is the Hamiltonian in the product space and $E = E_1 + E_2$ is the energy, respectively, of the two-particle system.)

   (a) $\Psi(x_1, x_2) = \psi_{n_1}(x_1) + \psi_{n_2}(x_2)$

   (b) $\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$

   (c) $\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)$

   (d) $\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)$

Consider the following conversation regarding whether the basis states for constructing the two-particle stationary state wavefunction for a system of two non-interacting identical particles can be written in terms of the sum of the single-particle wavefunctions.

Student 1: The basis states that can be used to construct a two-particle stationary state wavefunction for a system of two non-interacting identical particles can be written in terms of the sum of the single-particle wavefunctions, $\Psi(x_1, x_2) = \psi_{n_1}(x_1) + \psi_{n_2}(x_2)$.

Student 2: I disagree. The sum of the single-particle states $\psi_{n_1}(x_1)+\psi_{n_2}(x_2)$ is not in the Hilbert space of the two-particle system. When the two-particle Hamiltonian $\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2$ acts on the state $\psi_{n_1}(x_1)+\psi_{n_2}(x_2)$, there are inconsistencies. Consider terms of the type $\hat{\mathbf{H}}_{\mathbf{1}}\psi_{n_2}(x_2)$ when $\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2$ acts on $\psi_{n_1}(x_1)+\psi_{n_2}(x_2)$.

Student 1: Isn't $\hat{\mathbf{H}}_1\psi_{n_2}(x_2) = 0$?

Student 2: No. The single-particle Hamiltonian $\hat{H}_1$ only acts on the wavefunction corresponding to particle one. The wavefunction $\psi_{n_2}(x_2)$ can be written as $1 \cdot \psi_{n_2}(x_2)$. The wavefunction corresponding to particle one is "1", which is not normalizable.

Student 3: I agree with Student 2. The sum of the single-particle states $\psi_{n_1}(x_1) + \psi_{n_2}(x_2)$ cannot be a basis state for a two-particle system.

Explain why you agree or disagree with the students.

Consider the following conversation regarding whether the basis states for constructing the many-particle stationary state wavefunction for a system of two non-interacting identical particles can be written in terms of the product of the single-particle wavefunctions.
Student 1: The basis states used to construct a two-particle stationary state wavefunction for a system of two non-interacting identical particles can be written in terms of the product of the single-particle wavefunctions, such as $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$.
Student 2: I agree with Student 1. Also, if we consider terms of the type $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ in the wavefunction for a system of two non-interacting identical particles, then it satisfies the TISE, as follows:

$$
\begin{aligned}
\hat{\mathbf{H}}\psi_{n_1}(x_1)\psi_{n_2}(x_2) &= (\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= (\hat{H}_1 \otimes \hat{I}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) + (\hat{I}_1 \otimes \hat{H}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= [\hat{H}_1\psi_{n_1}(x_1)][\hat{I}_2\psi_{n_2}(x_2)] + [\hat{I}_1\psi_{n_1}(x_1)][\hat{H}_2\psi_{n_2}(x_2)] \\
&= [\hat{H}_1\psi_{n_1}(x_1)]\psi_{n_2}(x_2) + \psi_{n_2}(x_2)[\hat{H}_2\psi_{n_1}(x_1)] \\
&= E_{n_1}\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_1)E_{n_2}\psi_{n_2}(x_2) \\
&= E_{n_1}\psi_{n_1}(x_1)\psi_{n_2}(x_2) + E_{n_2}\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= (E_{n_1} + E_{n_2})\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= E\psi_{n_1}(x_1)\psi_{n_2}(x_2),
\end{aligned}
$$

in which $E = E_{n_1} + E_{n_2}$.

Explain why you agree or disagree with each student.

Consider the following conversation regarding whether the basis states consisting of the product of the single-particle stationary state wavefunctions span the product space of the many-particle system.
Student 1: The products of the single-particle stationary state wavefunctions are solutions to the TISE and therefore, they must be basis states for the system of N non-interacting identical particles.
Student 2: I agree. A complete set of energy eigenstates $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ will span the product space and will form a suitable basis.
Student 3: I agree with both Student 1 and Student 2. Since the products of the single-particle stationary state wavefunctions form a complete set of energy eigenstates for the many-particle system, they must span the product space for the many-particle system.

Explain why you agree or disagree with each student.

Summarize in your own words whether the sums or products of the single-particle wavefunctions can form a suitable basis for N non-interacting identical particles in the product space.

- The following conversation and questions will help you learn about the notation for the stationary state wavefunction for a system of N non-interacting identical particles

Consider the following conversation regarding whether the single-particle wavefunctions in the basis states should have the same or different coordinates to properly specify a three-particle wavefunction for a system of three non-interacting identical particles.

Student 1: We must assign a different coordinate to each identical particle. The wavefunction will have basis states such as $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$.

Student 2: No. I disagree with Student 1. When the particles are indistinguishable, we can't possibly distinguish their individual coordinates. So the wavefunction will have basis states such as $\psi_{n_1}(x)\psi_{n_2}(x)\psi_{n_3}(x)$.

Explain why you agree or disagree with each student.

5. After each statement, explain why you agree or disagree with the following students. If you disagree, write a correct statement.

   (a) **Student 1:** $\psi_{n_1}(x)\psi_{n_2}(x)$ is a basis state that can be used to construct the two-particle stationary state wavefunction for a system of two non-interacting particles. Particle 1 is in a single-particle state denoted by n_1 and particle 2 is in a single-particle state denoted by n_2.

   (b) **Student 2:** $\psi_{n_1}(x_2)\psi_{n_2}(x_1)$ is a basis state that can be used to construct the two-particle stationary state wavefunction for a system of two non-interacting particles. Particle 1 with coordinate x_2 is in a single-particle state denoted by n_1 and particle 2 with coordinate x_1 is in a single-particle state denoted by n_2.

   (c) **Student 3:** $\psi_{n_2}(x_2)\psi_{n_1}(x_1)$ is a basis state that can be used to construct the two-particle stationary state wavefunction for a system of two non-interacting particles. Particle 1 with coordinate x_1 is in a single-particle state denoted by n_1 and particle 2 with coordinate x_2 is in a single-particle state denoted by n_2.

   (d) **Student 4:** $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$ is a basis state that can be used to construct the three-particle stationary state wavefunction for a system of three non-interacting particles. Particle 1 with coordinate x_1 is in a single-particle state denoted by n_1, particle 2 with coordinate x_2 is in a single-particle state denoted by n_2, and particle 3 with coordinate x_3 is in a single-particle state denoted by n_3.

6. In your own words, describe what the symbols x_1, x_2, and x_3 in the basis state $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)$ mean to you. (Labels representing the single-particle states are n_1, n_2 and n_1, respectively, with two of the labels being the same.)

Consider the following conversation regarding whether a different ordering of the single-particle wavefunctions in the basis states yields a different basis state for a system of non-interacting identical particles.
Student 1: For a system of two non-interacting identical particles, the terms $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ and $\psi_{n_2}(x_2)\psi_{n_1}(x_1)$ represent two different basis states.
Student 2: No. I disagree with Student 1. When writing the basis states, different orderings of the single-particle wavefunctions does not produce a different basis state. Both terms $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ and $\psi_{n_2}(x_2)\psi_{n_1}(x_1)$ represent the same basis state in which particle 1 with coordinate x_1 is in a single-particle state denoted by n_1, and particle 2 with coordinate x_2 is in a single-particle state denoted by n_2.

Explain why you agree or disagree with each student.

**** CHECKPOINT: Check your answers to questions 2-6. ****

2. Student 1 is correct.
3a. There is an inconsistency in the term $\hat{\mathbf{H}}_1[\psi_{n_2}(x_2)]$. The single-particle Hamiltonian $\hat{H}_1$ can only act on the wavefunction in the part of the Hilbert space corresponding to particle 1 but this term has a wavefunction "1" corresponding to particle 1 which is not possible (in other words, $\hat{H}_1$ acts on "1" for the wavefunction which is not a possible wavefunction since it is not normalizable)
3b. There is an inconsistency in the term $\hat{\mathbf{H}}_2[\psi_{n_1}(x_1)]$. The single-particle Hamiltonian $\hat{H}_2$ can only act on the wavefunction in the part of the Hilbert space corresponding to particle 2 but this term has a wavefunction "1" corresponding to particle 2 which is not possible (in other words, $\hat{H}_1$ acts on "1" for the wavefunction which is not a possible wavefunction since it is not normalizable)
$\hat{\mathbf{H}}_2[\psi_{n_1}(x_1) + \psi_{n_2}(x_2)]$ is undefined as the term $\hat{\mathbf{H}}_2\psi_{n_1}(x_1) = (\hat{I}_1 \otimes \hat{H}_2)\psi_{n_1}(x_1) = [\hat{I}_1\psi_{n_1}(x_1)][\hat{H}_2 1]$ and 1 is not a normalizable wavefunction for particle 2.
3c. $(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)[\psi_{n_1}(x_1) + \psi_{n_2}(x_2)]$ is undefined by reasoning as in 3a and 3b.
3d. $\hat{\mathbf{H}}_1[\psi_{n_1}(x_1)\psi_{n_2}(x_2)] = E_{n_1}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]$
3e. $\hat{\mathbf{H}}_2[\psi_{n_1}(x_1)\psi_{n_2}(x_2)] = E_{n_2}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]$
3f.

$$
\begin{aligned}
\hat{\mathbf{H}}\psi_{n_1}(x_1)\psi_{n_2}(x_2) &= (\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= (\hat{H}_1 \otimes \hat{I}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) + (\hat{I}_1 \otimes \hat{H}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= \hat{H}_1\psi_{n_1}(x_1)\hat{I}_2\psi_{n_2}(x_2) + \hat{I}_1\psi_{n_1}(x_1)\hat{H}_2\psi_{n_2}(x_2) \\
&= \hat{H}_1\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_1)\hat{H}_2\psi_{n_2}(x_2) \\
&= [\hat{H}_1\psi_{n_1}(x_1)]\psi_{n_2}(x_2) + \psi_{n_1}(x_1)[\hat{H}_2\psi_{n_2}(x_2)] \\
&= E_{n_1}\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_1)E_{n_2}\psi_{n_2}(x_2) \\
&= E_{n_1}\psi_{n_1}(x_1)\psi_{n_2}(x_2) + E_{n_2}\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= (E_{n_1} + E_{n_2})\psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
&= E\psi_{n_1}(x_1)\psi_{n_2}(x_2)
\end{aligned}
$$

3g.

$$
\begin{aligned}
(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)] &= E_{n_1}\psi_{n_1}(x_1)\psi_{n_2}(x_2) + E_{n_2}\psi_{n_2}(x_1)\psi_{n_1}(x_2) \\
&\quad + E_{n_2}\psi_{n_1}(x_1)\psi_{n_2}(x_2) + E_{n_1}\psi_{n_2}(x_1)\psi_{n_1}(x_2) \\
&= (E_{n_1} + E_{n_2})[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)] \\
&= E[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]
\end{aligned}
$$

3h.

$$
\begin{aligned}
(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)] &= E_{n_1}\psi_{n_1}(x_1)\psi_{n_2}(x_2) - E_{n_2}\psi_{n_2}(x_1)\psi_{n_1}(x_2) \\
&\quad + E_{n_2}\psi_{n_1}(x_1)\psi_{n_2}(x_2) - E_{n_1}\psi_{n_2}(x_1)\psi_{n_1}(x_2) \\
&= (E_{n_1} + E_{n_2})[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)] \\
&= E[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)]
\end{aligned}
$$

4. b, c, and d. The wavefunctions in the preceding question 3f, 3g, and 3h, which are products of the single-particle wavefunctions, all satisfy the TISE for $\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$ and are possible many-particle stationary state wavefunctions.
5a. Student 1 is incorrect. The coordinates for each particle must be unique in the basis states (e.g., particle 1 has coordinate x_1 and particle 2 has coordinate x_2).
5b. Student 2 is incorrect. $\Psi(x_1, x_2) = \psi_{n_1}(x_2)\psi_{n_2}(x_1)$ is a basis state that can be used to construct the two-particle stationary state wavefunction for a system of two non-interacting particles. Particle 1 with coordinate x_1 is in a single-particle state denoted by n_2 and particle 2 with coordinate x_2 is in a single-particle state denoted by n_1.
5c. Student 3 is correct.
5d. Student 4 is correct.

6. For the system of three non-interacting particles, particle 1 with coordinate x_1 is in a single-particle state denoted by n_1, particle 2 with coordinate x_2 is in a single-particle state denoted by n_2, and particle 3 with coordinate x_3 is in a single-particle state denoted by n_1.

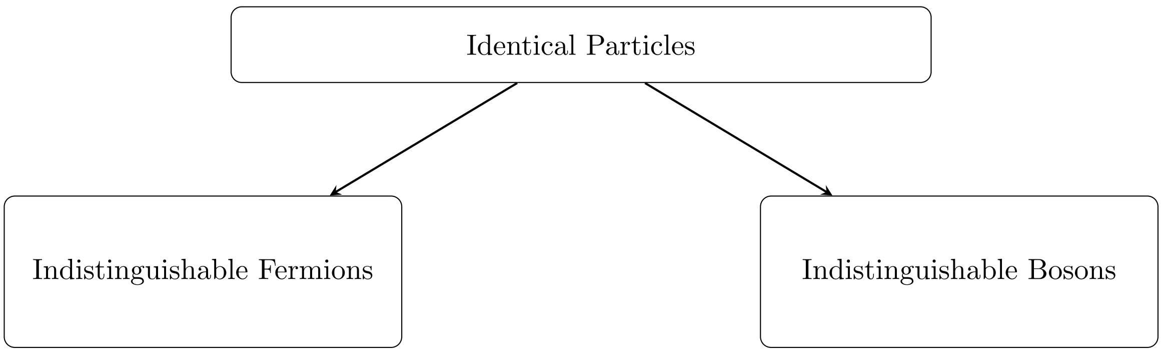
If your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

Summary of the Basis States for a System of N Non-Interacting Particles.

- The basis states used to construct the many-particle stationary state wavefunction for a system of N non-interacting identical particles are written in terms of products of the single-particle wavefunctions (NOT the sum of the single-particle wavefunctions) with different coordinates x_i for each particle.

3.3 Stationary State Wavefunction for a System of N Identical Particles which are Indistinguishable

- Now that we know that the products of the single-particles wavefunctions form appropriate basis states for the product space, let's focus on how to use these basis states to construct the many-particle stationary state wavefunction (i.e., the form of the many-particle stationary state wavefunction for identical particles which reflects indistinguishability).
- A system of identical particles which are indistinguishable can consist of either a system of identical fermions or identical bosons.



Consider the following conversation regarding identical particles which are indistinguisble.
Student 1: If we have two identical fermions, we can paint one fermion red and the other fermion green. Then, all we need to do is to keep track of the color to keep track of each fermion.
Student 2: In general, in quantum mechanics, if two particles in a system are identical fermions, we couldn't paint one red and the other green. Quantum particles are truly indistinguishable. There is no measurement we can perform that could distinguish one identical fermion from the other. For example, there is no measurement that can distinguish which fermion was in which single-particle state and had which coordinate. The wavefunction must reflect the fact that we cannot attach identifiers to each identical fermion.
Student 3: Yes. Similarly, if both particles are identical bosons, we couldn't paint one red and the other green either. In general, when the single-particle wavefunctions for the two identical bosons overlap, there is no measurement we can perform that could distinguish one boson from the other, for example, which boson had which coordinate and was in which single-particle state.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the spin of identical particles regardless of whether the particles are fundamental particles (indivisible or composite).
Student 1: When we have a system of identical particles, all particles have the same intrinsic properties such as mass, charge, and spin.
Student 2: I agree. Also, the property of spin differentiates a boson from a fermion. The spin of a boson must be an integer. For example, Helium-4 is a boson since it has integer spin. The spin of a fermion must be a half-integer. For example, an electron, proton, and neutron are fermions with spin 1/2.

Explain why you agree or disagree with the students.

Consider the following conversation regarding whether the coordinates of each particle should be the same or different in the wavefunction for a system of non-interacting identical particles which are indistinguishable.

Student 1: For a system of three identical particles, the wavefunction will have terms such as $\psi_{n_1}(x)\psi_{n_2}(x)\psi_{n_3}(x)$ in which $\psi_{n_1}(x), \psi_{n_2}(x)$, and $\psi_{n_3}(x)$ are the single-particle wavefunctions with the same coordinate for all three particles since the particles are indistinguishable.

Student 2: I disagree. Even though the particles are indistinguishable, we must still assign a different coordinate to each particle in a given state. The wavefunction will have terms such as $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$.

Student 3: No. I agree with Student 1 and disagree with Student 2. When the particles are indistinguishable, we can't possibly distinguish their individual coordinates. So the wavefunction will have terms such as $\psi_{n_1}(x)\psi_{n_2}(x)\psi_{n_3}(x)$.

Explain why you agree or disagree with each student.

Student 2 is correct in the preceding conversation.

- The coordinates do not account for the indistinguishability of the particles, rather the indistinguishability is reflected in the way the many-particle wavefunction is written (either as a completely symmetric or antisymmetric wavefunction).
- The wavefunction for indistinguishable fermions has different properties than the wavefunction for indistinguishable bosons.
- Before considering the wavefunction for indistinguishable fermions or indistinguishable bosons, let's review how to determine whether a many-particle wavefunction is completely symmetric versus antisymmetric with respect to the exchange of any two particles.

Symmetric Wavefunction: A **symmetric wavefunction** of two-particles $\Psi(x_1, x_2)$ produces the same wavefunction (with the same sign) when the two particles are exchanged. Therefore,

$$\Psi(x_2, x_1) = \Psi(x_1, x_2).$$

A completely **symmetric wavefunction** for N particles $\Psi(x_1, x_2, x_3 \ldots, x_i, \ldots, x_j, \ldots, x_N)$ produces the same wavefunction (with the same sign) when any two particles labeled by x_i and x_j are exchanged:

$$\Psi(x_1, x_2, x_3 \ldots, x_j, \ldots, x_i, \ldots, x_N) = \Psi(x_1, x_2, x_3 \ldots, x_i, \ldots, x_j, \ldots, x_N).$$

The following permutations of coordinates of particles (underlined) are all examples of the consequences of exchanging particles for a completely symmetric wavefunction (i.e., the many-particle wavefunction is unchanged)

i. One permutation

$$\Psi(\underline{x_1}, \underline{x_2}, x_3, \ldots, x_N) = \Psi(\underline{x_2}, \underline{x_1}, x_3, \ldots, x_N) \quad \text{(Permuting } x_1 \text{ and } x_2\text{)}$$

ii. Two total permutations

$$\begin{aligned} \Psi(x_1, x_2, x_3, x_3, x_4, \ldots, x_N) &= \Psi(\underline{x_2}, \underline{x_1}, x_3, x_4, \ldots, x_N) \quad \text{(First permutation: Permuting } x_1 \text{ and } x_2\text{)} \\ &= \Psi(x_2, \underline{x_3}, \underline{x_1}, x_4, \ldots, x_N) \quad \text{(Second permutation: Permuting } x_1 \text{ and } x_3\text{)} \end{aligned}$$

iii. Three total permuations

$$\begin{aligned} \Psi(x_1, x_2, x_3, x_4, \ldots, x_N) &= \Psi(\underline{x_2}, \underline{x_1}, x_3, x_4, \ldots, x_N) \quad \text{(First permutation: Permuting } x_1 \text{ and } x_2\text{)} \\ &= \Psi(x_2, \underline{x_3}, \underline{x_1}, x_4, \ldots, x_N) \quad \text{(Second permutation: Permuting } x_1 \text{ and } x_3\text{)} \\ &= \Psi(\underline{x_3}, \underline{x_2}, x_1, x_4, \ldots, x_N) \quad \text{(Third permutation: Permuting } x_2 \text{ and } x_3\text{)} \end{aligned}$$

Continuing in this manner, you can perform any number of permutations to show that the many-particle is unchanged for each exchange of particles.

- The wavefunction for indistinguishable bosons must be a completely symmetric wavefunction with respect to exchange of any two particles.

Antisymmetric Wavefunction: An **antisymmetric wavefunction** of two-particles $\Psi(x_1, x_2)$ produces a wavefunction that is related to the original wavefunction as follows when the two particles are exchanged:

$$\Psi(x_2, x_1) = -\Psi(x_1, x_2).$$

A completely **antisymmetric wavefunction** of N particles $\Psi(x_1, x_2, x_3, \ldots, x_N)$ produces a wavefunction that is related to the original wavefunction as follows when two particles are exchanged. The following permutations of the coordinates are all examples of the consequences of exchanging particles for a completely antisymmetric wavefunction
i. One permutation

$$\Psi(\underline{x_1}, \underline{x_2}, x_3, \ldots, x_N) = -\Psi(\underline{x_2}, \underline{x_1}, x_3, \ldots, x_N) \quad \text{(Permuting } x_1 \text{ and } x_2\text{)}$$

ii. Two total permutations

$$\begin{aligned}
\Psi(x_1, x_2, x_3, x_4, \ldots, x_N) &= -\Psi(\underline{x_2}, \underline{x_1}, x_3, x_4, \ldots, x_N) && \text{(First Permutation: Permuting } x_1 \text{ and } x_2\text{)} \\
&= -[-\Psi(x_2, \underline{x_3}, \underline{x_1}, x_4, \ldots, x_N)] && \text{(Second Permutation: Permuting } x_1 \text{ and } x_3\text{)} \\
&= \Psi(x_2, x_3, x_1, x_4, \ldots, x_N) && \text{(Simplifying } -1 \times -1 \text{ for two permutations)}
\end{aligned}$$

iii. Three total permutations

$$\begin{aligned}
\Psi(x_1, x_2, x_3, x_4, \ldots, x_N) &= -\Psi(\underline{x_2}, \underline{x_1}, x_3, x_4, \ldots, x_N) && \text{(First Permutation: Permuting } x_1 \text{ and } x_2\text{)} \\
&= -[-\Psi(x_2, \underline{x_3}, \underline{x_1}, x_4, \ldots, x_N)] && \text{(Second Permutation: Permuting } x_1 \text{ and } x_3\text{)} \\
&= \Psi(x_2, x_3, x_1, x_4, \ldots, x_N) && \text{(Simplifying } -1 \times -1 \text{ for two permutations)} \\
&= -\Psi(\underline{x_3}, \underline{x_2}, x_1, x_4, \ldots, x_N) && \text{(Third Permuation: Permuting } x_2 \text{ and } x_3\text{)}
\end{aligned}$$

Continuing in this manner, you can perform any number of permutations to show that the many-particle wavefunction develops a plus or minus sign for each exchange of particles depending upon whether the number of exchanges was even or odd, respectively.

- The wavefunction for indistinguishable fermions must be a completely antisymmetric wavefunction with respect to the exchange of any two particles.

Consider the following conversation regarding the only two ways of constructing a wavefunction for identical particles which are indistinguishable (either completely symmetric or completely antisymmetric with respect to exchange of any two particles).
Student 1: Since there is no measurement we can perform to distinguish different identical particles in a system consisting of N identical particles, the wavefunction must reflect this symmetry.
Student 2: I agree with Student 1. There are two possible ways to construct the wavefunction for a system of N non-interacting indistinguishable particles from the single-particle wavefunctions for that system. The wavefunction could be either completely symmetric or completely antisymmetric with respect to exchange of two particles because it is $|\psi|^2$ that determines the measurable properties and the overall sign of the many-particle wavefunction is not important.

Explain why you agree or disagree with the students.

Consider the following conversation regarding the eigenvalues of the "permutation operator."
Student 1: Let's consider the permutation operator $\hat{P}_{ij}$ acting on a many-particle stationary state wavefunction for a system of identical particles. The permutation operator $\hat{P}_{ij}$ acting on the many-particle stationary state wavefunction exchanges particle i and particle j in the many-particle stationary state wavefunction.
Student 2: I agree. If the permutation operator $\hat{P}_{ij}$ is applied twice, the original wavefunction is obtained. That is,

$$\hat{P}_{ij}^2 \Psi(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_N) = \Psi(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_N).$$

Therefore, $\hat{P}_{ij}^2 = \hat{I}$, in which $\hat{I}$ is the identity operator. Thus, the eigenvalues of the permutation operator $\hat{P}_{ij}$ are ± 1. The eigenvalue 1 corresponds to the completely symmetric bosonic wavefunction and the eigenvalue -1 corresponds to the completely antisymmetric fermionic wavefunction.

Explain why you agree or disagree with the students.

3.3.1 Stationary State Wavefunction for a System of N Indistinguishable Fermions

- Now let's consider the case in which the identical particles are indistinguishable fermions.
- We will begin with a system of two fermions and then consider a system of three fermions and finally consider a system of N fermions.

7. Consider a system of two non-interacting identical fermions in which $\psi_{n_1}(x)$ and $\psi_{n_2}(x)$ are the single-particle wavefunctions for the system and $n_1 \neq n_2$. Choose all of the following normalized wavefunctions that are appropriate for a system of two non-interacting fermions considering that indistinguishable fermions must have a completely antisymmetric wavefunction.

   (a) $\psi_{n_1}(x_1)\psi_{n_2}(x_1)$ (same coordinate)

   (b) $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$

   (c) $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$

   (d) $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$

   (e) $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ (same state label n_1)

Consider the following conversation regarding the wavefunction for a system of two non-interacting indistinguishable fermions.

Student 1: For a system of two non-interacting indistinguishable fermions, the wavefunction describing the system is $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, in which $\psi_{n_1}(x_1)$ and $\psi_{n_2}(x_2)$ are the single-particle wavefunctions for the two-particles.

Student 2: I disagree. If the system consists of two fermions, there is no way to distinguish which fermion is in the state labeled by n_1 and which is in the state labeled by n_2. The wavefunction must reflect this symmetry.

Student 3: I agree with Student 2. The wavefunction describing a system of non-interacting indistinguishable fermions must be completely antisymmetric. Therefore, the normalized wavefunction for a system of two non-interacting fermions must be $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$.[1]

Explain why you agree or disagree with each student.

[1] The wavefunction for a system of indistinguishable fermions must always be completely antisymmetric. This must also be true when the system includes interactions between the indistinguishable fermions so that the stationary state wavefunction cannot be expressed as $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$.

Consider the following conversation regarding whether the Pauli exclusion principle and identical fermions having a completely antisymmetric wavefunction are consistent with each other.
Student 1: The fact that a wavefunction for a system of fermions must be completely antisymmetric is consistent with the Pauli exclusion principle.
Student 2: I thought the Pauli exclusion principle states that no two fermions can be in the same single-particle state. How is that consistent with the wavefunction being completely antisymmetric?
Student 1: Let's suppose we have two fermions in the same single-particle state. Then $n_1 = n_2$ and the wavefunction would be $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2) - \psi_{n_1}(x_2)\psi_{n_1}(x_1)] = 0$. Thus $\Psi(x_1, x_2) = 0$ is not a possible wavefunction.
Student 3: The same is true for a system of more than two indistinguishable fermions. Since a system of fermions has a completely antisymmetric wavefunction, no two fermions can be in the same single-particle state. If you try to put two or more fermions in the same state, the wavefunction will be zero for the N-fermion system.

Explain why you agree or disagree with Student 1 and Student 3.

Consider the following conversation regarding whether different orderings of the single-particle stationary state wavefunctions yield different many-particle wavefunctions.
Student 1: The basis states for a system of non-interacting identical fermions with only two available single-particle states n_1 and n_2 are $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, $\psi_{n_2}(x_2)\psi_{n_1}(x_1)$, $\psi_{n_2}(x_1)\psi_{n_1}(x_2)$, and $\psi_{n_1}(x_2)\psi_{n_2}(x_1)$. The normalized many-particle stationary state wavefunction for a system of two indistinguishable fermions is $\frac{1}{\sqrt{4}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_2)\psi_{n_1}(x_1) - \psi_{n_2}(x_1)\psi_{n_1}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$.
Student 2: I disagree with Student 1. The terms $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ and $\psi_{n_2}(x_2)\psi_{n_1}(x_1)$ represent two ways to write the same basis state. Changing the order of the single-particle wavefunctions does not give a different basis state.
Student 3: I agree with Student 2. The expression $\frac{1}{\sqrt{4}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)+\psi_{n_2}(x_2)\psi_{n_1}(x_1)-\psi_{n_2}(x_1)\psi_{n_1}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)] = \frac{1}{\sqrt{4}}[2\psi_{n_1}(x_1)\psi_{n_2}(x_2) - 2\psi_{n_2}(x_1)\psi_{n_1}(x_2)] = \psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)$, which is not a properly normalized wavefunction. The normalization factor should be $\frac{1}{\sqrt{2}}$.

Explain why you agree or disagree with Student 1 and Student 3.

8. Is the completely antisymmetric wavefunction $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ a stationary state wavefunction for the two-fermion system? Explain.

Consider the following conversation regarding whether after antisymmetrizing the wavefunction for a system of two non-interacting fermions, the state remains a stationary state wavefunction of the many-particle system with $n_1 \neq n_2$.

Student 1: When we completely antisymmetrize the wavefunction for two fermions, the wavefunction is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$. However, since this is a linear superposition of two basis states, it is not a stationary state wavefunction for the two-particle system.

Student 2: I disagree with Student 1's claim that $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ is not a stationary state wavefunction for the two-particle system. If we completely antisymmetrize the wavefunction for a system of two non-interacting fermions, then this completely antisymmetric wavefunction $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ constructed from products of single-particle wavefunctions is a stationary state wavefunction for the two-particle system. That is,

$$\begin{aligned}
\hat{\mathbf{H}}\Psi(x_1, x_2) &= \hat{\mathbf{H}}\left\{\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\} \\
&= E_1\left\{\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\} + E_2\left\{\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\} \\
&= E\left\{\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\} \\
&= E\Psi(x_1, x_2).
\end{aligned}$$

This is true because each basis state in the product space satisfies the TISE with the same energy $E = E_1 + E_2$

Explain why you agree or disagree with each student.

Use the following questions to check your answer to the preceding question about the conversation.

9. Consider a system of two non-interacting identical fermions. As we learned, the Hamiltonian for a system of two non-interacting identical particles is given by $\hat{\mathbf{H}} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$. Using the TISE, determine whether the completely antisymmetric wavefunction $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ is a stationary state wavefunction for the two fermion system.

   $\hat{\mathbf{H}}\Psi(x_1, x_2) = \hat{\mathbf{H}}\{\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\} =$

10. What is the energy for a system of two non-interacting identical fermions in which one fermion is in a single-particle state labeled by n_1 with energy E_{n_1} and the other fermion is in a single-particle state labeled by n_2 with energy E_{n_2}?

- Now, let's construct the completely antisymmetric wavefunction for a system of more than one non-interacting, indistinguishable fermion.
- We will begin with a system of two indistinguishable fermions followed by a system of three indistinguishable fermions.

11. Starting with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, construct the completely antisymmetric wavefunction for a system of two non-interacting, indistinguishable fermions by permuting the <u>coordinates</u> (hold n_1 and n_2 fixed) and combining the terms with different permutations to make the wavefunction completely antisymmetric.

12. Starting with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, construct the completely antisymmetric wavefunction for a system of two non-interacting, indistinguishable fermions by permuting the <u>labels n_1 and n_2 for the states</u> (hold x_1 and x_2 fixed) and combining the terms with different permutations to make the wavefunction completely antisymmetric.

13. Compare your answers to questions 11 and 12 and state the reasoning for what you found.

Consider the following conversation regarding constructing a completely antisymmetric wavefunction for a system of two indistinguishable fermions starting with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$.
Student 1: If we start with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, we can construct a completely antisymmetric wavefunction by **interchanging the two single-particle wavefunction labels**, multiplying the new permutation by -1 and then summing over all the permutations, which in this case is just two permutation. If we permute $\underline{n_1}$ and $\underline{n_2}$ in $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, the new term is $-\psi_{n_2}(x_1)\psi_{n_1}(x_2)$. After normalization, the completely antisymmetric wavefunction for a system of two identical fermions is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$.
Student 2: If we start with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, we can construct a completely antisymmetric wavefunction by **interchanging the coordinates**, multiplying the new permutation by -1 and then summing over all the permutations, which in this case is just two permutation. If we permute $\underline{x_1}$ and $\underline{x_2}$ in $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, the new term is $-\psi_{n_1}(x_2)\psi_{n_2}(x_1)$. The sum of the terms after normalization for the completely antisymmetric wavefunction for a system of two identical fermions is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$.
Student 3: I agree with both Student 1 and Student 2. Both students constructed the same completely antisymmetric wavefunction. The single-particle wavefunctions are not operators, so we can switch the order of single-particle wavefunctions, i.e., $\psi_{n_1}(x_2)\psi_{n_2}(x_1) = \psi_{n_2}(x_1)\psi_{n_1}(x_2)$. The completely antisymmetric wavefunction can be generated by interchanging either the coordinates or the labels for the states.

Explain why you agree or disagree with the students.

Consider the following conversation regarding constructing a completely antisymmetric wavefunction for a system of indistinguishable fermions by switching both the coordinates and the labels for the states.
Student 1: If we interchange **both** the labels for the states and the coordinates, the resulting wavefunction is a completely antisymmetric wavefunction for the system of identical fermions.
Student 2: I disagree with Student 1. Let's consider a system of two indistinguishable fermions. If we start with the basis state $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ and interchange two single-particle wavefunction labels and multiply the new permutation by -1, the new term is $-\psi_{n_2}(x_1)\psi_{n_1}(x_2)$. Now if we interchange the coordinates of the two-particles and multiply the new permutation by -1, the new term is $\psi_{n_2}(x_2)\psi_{n_1}(x_1) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$. By switching both the coordinates and the labels, we recovered the original expression and did not generate a new term. The original expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ that we got back by exchanging both the labels for the states and the coordinates is not antisymmetric and therefore it cannot be the wavefunction for a system of two indistinguishable fermions.
Student 3: I agree with Student 2. For a system of indistinguishable fermions, we cannot generate a completely antisymmetric wavefunction by switching **both** the coordinates and the labels for the states. We should only permute one of them to generate a completely antisymmetric wavefunction.

Explain why you agree or disagree with each student.

14. Starting with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$, construct the completely antisymmetric wavefunction for the system of three indistinguishable fermions. Hint: Switch either the coordinates or the states (but not both) two at a time and remember to make the wavefunction completely antisymmetric by multiplying the new permutation by -1 each time you interchange two particles. Two interchanges will produce -1 $\times$ -1 =1 times the new permutation. Then sum all of the permutations and normalize the completely antisymmetric wavefunction.

Consider the following conversation regarding the number of terms and the normalization factor for a completely antisymmetric wavefunction for a system of indistinguishable fermions.
Student 1: When constructing the completely antisymmetric wavefunction for a system of three indistinguishable fermions, how do I know that I have found all the possible permutations?
Student 2: In general, for a system of N indistinguishable fermions, there are $N!$ permutations of the labels. For example, there are $N!$ permutations of the coordinates $x_1, x_2, \ldots, x_N$ or $N!$ permutations of the labels for the single-particle states $\psi_{n_1}, \psi_{n_2}, \ldots, \psi_{n_N}$. The normalization factor is $\frac{1}{\sqrt{N!}}$.
Student 3: I agree with Student 2. For a system of three indistinguishable fermions, the completely antisymmetric wavefunction will have $3! = 6$ terms and the normalization factor will be $\frac{1}{\sqrt{6}}$.

Explain why you agree or disagree with Student 2 and Student 3.

Consider the following conversation regarding a method for constructing completely antisymmetric wavefunctions for indistinguishable fermions.
Student 1: To find the completely antisymmetric wavefunction for a system of three indistinguishable fermions, we start with the expression $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$ and then find all possible permutations of either the coordinates (x_1, x_2, x_3) or the state indices (n_1, n_2, n_3). Each time we interchange two labels, we multiply the new permuted term by -1. Once we find all the permutations, we add them and normalize the completely antisymmetric wavefunction obtained.
Student 2: Although I agree with Student 1's method for more than two-particles, it can be easy to make a mistake with the sign of each term or omit a term altogether. A more systematic approach to help eliminate these sign mistakes is to use the "Slater determinant". For three-particles, the Slater determinant is

$$A\begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2) \\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3) \end{vmatrix} = \begin{array}{l} A[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3) \\ +\psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3) + \psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) - \psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)] \end{array}$$

in which A is the normalization constant which needs to be found separately. Here, $A = \frac{1}{\sqrt{N!}} = \frac{1}{\sqrt{6}}$ for a system of three fermions since each single-particle state is itself normalized. The Slater determinant can equivalently be expressed as

$$A\begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_1}(x_2) & \psi_{n_1}(x_3) \\ \psi_{n_2}(x_1) & \psi_{n_2}(x_2) & \psi_{n_2}(x_3) \\ \psi_{n_3}(x_1) & \psi_{n_3}(x_2) & \psi_{n_3}(x_3) \end{vmatrix} = \begin{array}{l} A[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) \\ +\psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1) + \psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2) - \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)]. \end{array}$$

Student 3: I agree with Student 2. The wavefunction is the same using either form of the Slater determinant since the rows and columns are transposed. Also, the Slater determinant works for a system of any number of fermions although even this method can become tedious when applied to more than three fermions.

Explain why you agree or disagree with the students.

15. Using the Slater determinant, determine the stationary state wavefunction of a system of two fermions and check your answer to question 7.

Consider the following conversation regarding the Slater determinant and the Pauli exclusion principle for a system of two identical fermions.

Student 1: The Slater determinant yields a many-particle wavefunction which is consistent with the Pauli exclusion principle. For example, for a system of two fermions, if we put both fermions in the same single-particle state, then

$$\begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_1}(x_2) \\ \psi_{n_1}(x_1) & \psi_{n_1}(x_2) \end{vmatrix} = \psi_{n_1}(x_1)\psi_{n_1}(x_2) - \psi_{n_1}(x_2)\psi_{n_1}(x_1) = 0,$$

which is not be a possible wavefunction since zero represents the absence of a wavefunction.

Student 2: I agree with Student 1. We can extend the Slater determinant method to find the many-particle wavefunction for a system with more than two particles. Consistent with Pauli's exclusion principle, having two particles in the same single-particle state produces two columns or rows with the same entries so the Slater determinant of the many-particle wavefunction is zero, which is not a possible wavefunction.

Explain why you agree or disagree with the students.

****CHECKPOINT: Check your answers to questions 7-14. ****

7. d
8. Yes, the completely antisymmetric wavefunction $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ is a stationary state wavefunction for the two fermion system as it satisfies the TISE, $\hat{H}\Psi(x_1,x_2)=E\Psi(x_1,x_2)$
9.

$$\begin{aligned}
\hat{\mathbf{H}}\Psi(x_1,x_2) &= (\hat{\mathbf{H}}_1+\hat{\mathbf{H}}_2)\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&= \hat{\mathbf{H}}_1\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\}-\hat{\mathbf{H}}_1\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&\quad +\hat{\mathbf{H}}_2\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\}-\hat{\mathbf{H}}_2\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&= E_{n_1}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\}-E_{n_2}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&\quad +E_{n_2}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\}-E_{n_1}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&= E_{n_1}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&\quad +E_{n_2}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\}\\
&= (E_{n_1}+E_{n_2})\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\}\\
&= E\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)]\right\} \qquad \text{in which } E=E_{n_1}+E_{n_2}\\
&= E\Psi(x_1,x_2)
\end{aligned}$$

10. $E=E_{n_1}+E_{n_2}$
11. $\Psi(x_1,x_2)=\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_1}(x_2)\psi_{n_2}(x_1)]$
12. $\Psi(x_1,x_2)=\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_2}(x_1)\psi_{n_1}(x_2)]$
13. The completely antisymmetric wavefunction for the system of two fermions is the same if we permute either the coordinates or the labels for the states (but NOT both simultaneously).
14.

Permutation	Switch	New Permutation
$\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$	$n_1 \leftrightarrow n_2$	$-\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3)$
$-\psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3)$	$n_1 \leftrightarrow n_3$	$\psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$
$\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$	$n_2 \leftrightarrow n_3$	$-\psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3)$
$-\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3)$	$n_1 \leftrightarrow n_3$	$\psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3)$
$\psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$	$n_1 \leftrightarrow n_2$	$-\psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)$

Adding the different permutations, we get the completely antisymmetric wavefunction

$$\Psi(x_1,x_2,x_3)=\frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)-\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_3}(x_3)+\psi_{n_3}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$$

$$-\psi_{n_1}(x_1)\psi_{n_3}(x_2)\psi_{n_2}(x_3)+\psi_{n_2}(x_1)\psi_{n_3}(x_2)\psi_{n_1}(x_3)-\psi_{n_3}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)]$$

15. $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)-\psi_{n_2}(x_1)\psi_{n_1}(x_2)]$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

Summary for the Properties of the Wavefunction for Fermions

- The wavefunction for a system of indistinguishable fermions is completely antisymmetric with respect to exchange of any two particles.

3.3.2 Stationary State Wavefunction for a System of N Indistinguishable Bosons

- Now let's consider the case in which the particles are indistinguishable bosons.

16. Consider a system of two non-interacting, indistinguishable bosons in which $\psi_{n_1}(x)$ and $\psi_{n_2}(x)$ are the single-particle wavefunctions for the system ($n_1 \neq n_2$). Choose all of the following wavefunctions that are appropriate for a system of two non-interacting indistinguishable bosons considering that bosons must have a completely symmetric wavefunction.

    (a) $\psi_{n_1}(x_1)\psi_{n_2}(x_1)$

    (b) $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$

    (c) $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$

    (d) $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$

    (e) $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ (same label n_1 for the states)

    Consider the following conversation regarding the wavefunction for a system of two non-interacting indistinguishable bosons.

    Student 1: For a system of two non-interacting, indistinguishable bosons, if the bosons are in the same single-particle state, say ψ_{n_1}, the wavefunction describing the two-particle system is $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$.

    Student 2: I disagree. If the system consists of two indistinguishable bosons, the bosons cannot be in the same single-particle state. So, $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ is not a possible wavefunction for a system of two non-interacting, indistinguishable bosons. $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ is the wavefunction for distinguishable particles only.

    Explain why you agree or disagree with each student.

    Consider the following conversation regarding two indistinguishable bosons in the same single-particle state.

    Student 1: If we have a system consisting of two indistinguishable bosons, then the Pauli exclusion principle tells us that the bosons must be in different single-particle states.

    Student 2: I disagree with Student 1. The Pauli exclusion principle applies only to fermions. Since the wavefunction for a system of indistinguishable bosons is symmetric with respect to exchange of two particles, the wavefunction is not zero when the indistinguishable bosons are in the same single-particle state.

    Student 3: I agree with Student 2. The antisymmetrized wavefunction for two indistinguishable fermions in the same single-particle state is zero, which is not a possible wavefunction consistent with Pauli's exclusion principle. However, for two indistinguishable bosons, if both bosons are in state n_1, then the normalized two-particle wavefunction would be $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$.

    Explain why you agree or disagree with each student.

Consider the following conversation regarding two indistinguishable bosons having the same two-particle stationary state wavefunction as a system of identical particles that can be treated as distinguishable.
Student 1: For two indistinguishable bosons, if both bosons are in state n_1, then the normalized two-particle wavefunction is $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$.
Student 2: I disagree with Student 1. The wavefunction $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ is not a possible stationary state wavefunction for a system of bosons. The wavefunction for a system of indistinguishable bosons must be completely symmetric and we must have a sum of terms in the wavefunction for it to be completely symmetric. The wavefunction $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ is only possible for a system of identical particles that can be treated as distinguishable.
Student 3: I agree with Student 1 and disagree with Student 2. A completely symmetric wavefunction does not necessarily have to be written in terms of a sum. The wavefunction $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ is completely symmetric with respect to exchange of the two particles. If all of the indistinguishable bosons are in the same single-particle state, then the many-particle wavefunction for a system of indistinguishable bosons is the same as the wavefunction for a system of identical particles that can be treated as distinguishable.

Explain why you agree or disagree with each student.

17. Check whether the wavefunction $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ satisfies the TISE and is symmetric with respect to exhange of the two particles.

Consider the following conversation regarding the wavefunction for a system of two non-interacting indistinguishable bosons when $n_1 \neq n_2$.
Student 1: For a system of two non-interacting indistinguishable bosons, if the two bosons are in different single-particle states, the wavefunction describing the two-particle system is $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, in which $\psi_{n_1}(x_1)$ and $\psi_{n_2}(x_2)$ are the single-particle wavefunctions for the two-particles.
Student 2: I disagree. If the system consists of two bosons, there is no way to distinguish which boson is in the single-particle state denoted by n_1 and which is in the single-particle state denoted by n_2. The wavefunction must reflect this symmetry.
Student 3: The wavefunction describing a system of non-interacting indistinguishable bosons must be completely symmetric.[2] Therefore, the two-particle wavefunction for a system of two non-interacting, indistinguishable bosons, where the bosons are in different single-particle states, must be $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)+\psi_{n_2}(x_1)\psi_{n_1}(x_2)]$.

Explain why you agree or disagree with each student.

[2]The wavefunction for a system of indistinguishable bosons must always be completely symmetric. This must also be true when the system includes interactions between the indistinguishable bosons so that the stationary state wavefunction cannot be expressed as $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)+\psi_{n_2}(x_1)\psi_{n_1}(x_2)]$.

18. Does the two-particle wavefunction $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$ satisfy the TISE for a two-particle system? Explain.

19. What is the energy for a system of two non-interacting identical bosons in which one boson is in a single-particle state labeled by n_1 and the other boson is in a single-particle state labeled by n_2?

20. For a system of <u>two</u> non-interacting, indistinguishable bosons, how many terms will be present in the two-particle wavefunction for the system if the bosons are in <u>different single-particle states</u>?

21. For a system of <u>two</u> non-interacting, indistinguishable bosons, how many terms will be present in the two-particle wavefunction for the system if the bosons are in the <u>same single-particle state</u>?

22. For a system of <u>three</u> non-interacting, indistinguishable bosons, how many terms will be present in the three-particle wavefunction for the system if <u>two of the three bosons are in the same single-particle stationary state</u>?

Consider the following conversation regarding the normalization factor for a system of indistinguishable bosons.

Student 1: For a system of N non-interacting, indistinguishable bosons, the normalization factor must be $\frac{1}{\sqrt{N!}}$.

Student 2: I agree with Student 1. To ensure we have a symmetric wavefunction, the many-particle wavefunction will be the sum of all the permutations of the product of the single-particle wavefunctions. Since there are $N!$ ways to permute the N single-particle wavefunctions, the normalization factor will be $\frac{1}{\sqrt{N!}}$.

Student 3: I disagree with both Student 1 and Student 2. The normalization factor will be $\frac{1}{\sqrt{N!}}$ only if all the bosons are in different single-particle states. If we have all of the bosons in one single-particle state, $\prod_{i=1}^{N} \psi_n(x_i)$ is a valid many-particle state, e.g., $\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$ is an appropriately symmetrized wavefunction and the overall normalization factor for $\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$ is 1 since all three particles are in the same single-particle state given by the label n_1. We must be careful not to over count the number of unique permutations of the N single-particle states.

Explain why you agree or disagree with each student.

23. Construct the completely symmetric normalized three-particle wavefunction for the system of three non-interacting, indistinguishable bosons in the following cases:

    (a) All the bosons are in different states.

    (b) Two of the bosons are in the same state ψ_{n_1}.

    (c) All the bosons are in the same state ψ_{n_1}.

****CHECKPOINT: Check your answers to questions 16-23. ****

16. c and e
17. Yes, the wavefunction $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ is symmetric with respect to exchange of the two particles and satisfies the TISE.

$$\hat{\mathbf{H}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)] = (E_{n_1} + E_{n_2})[\psi_{n_1}(x_1)\psi_{n_2}(x_2)] = E[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]$$

18. Yes, the completely symmetric wavefunction $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ is a stationary state wavefunction for the two boson system as it satisfies the TISE, $\hat{H}\Psi(x_1, x_2) = E\Psi(x_1, x_2)$.

$$\begin{aligned}
\hat{\mathbf{H}}\Psi(x_1,x_2) &= (\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&= \hat{\mathbf{H}}_1\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\} + \hat{\mathbf{H}}_1\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&\quad + \hat{\mathbf{H}}_2\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\} + \hat{\mathbf{H}}_2\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&= E_{n_1}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\} + E_{n_2}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&\quad + E_{n_2}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)]\right\} + E_{n_1}\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&= (E_{n_1} + E_{n_2})\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&= E\left\{\tfrac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]\right\} \\
&= E\Psi(x_1,x_2)
\end{aligned}$$

19 $E = E_{n_1} + E_{n_2}$
20. There must be two terms to satisfy the symmetrization requirement for bosons.
$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$
21. One. For example, if both bosons are in the single-particle state ψ_{n_1}, the many-particle stationary state wavefunction is $\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_1}(x_2)$
22. There must be three terms to satisfy the symmetrization requirement for bosons. For example, if two of the three bosons are in the single-particle state ψ_{n_1}, the many-particle stationary state wavefunction is $\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)]$
23a.

$$\begin{aligned}
\Psi(x_1,x_2,x_3) &= \tfrac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) \\
&\quad + \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1) + \psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2) + \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)]
\end{aligned}$$

23b. $\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)]$
23c. $\Psi(x_1, x_2, x_3) = \psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

Summary of Properties of the Wavefunction for Bosons

- The wavefunction for a system of indistinguishable bosons is completely symmetric with respect to exchange of any two particles.

3.3.3 Hypothetical Case: Stationary State Wavefunction for a System of N Non-Interacting Identical Particles if They Could Be Treated as Distinguishable

- Let's contrast the cases of indistinguishable fermions and indistinguishable bosons with a hypothetical case in which the identical particles could be treated as distinguishable.
- We compare the resulting many-particle stationary state wavefunctions to what was obtained for indistinguishable fermions and indistinguishable bosons to learn why care must be taken to ensure that the many-particle wavefunction reflects the indistinguishability of the particles.
- If identical particles (particles of one type with the same properties) could be treated as **distinguishable**, we can assign a distinct label (e.g., red, blue, etc.) to distinguish each particle from the other particles in the system even though the particles have the same properties.

Consider the following conversation regarding the symmetrization requirements of the wavefunction for a system of two non-interacting identical particles if they could be treated as **distinguishable**.
Student 1: For a system of two non-interacting identical particles which can be treated as distinguishable, we must still symmetrize the wavefunction.
Student 2: I disagree with Student 1. Since the particles can be treated as distinguishable, we can determine which particle is in which single-particle state. There is no requirement to symmetrize the wavefunction.

Explain why you agree or disagree with the students.

24. Consider a system of two non-interacting, identical particles which can be treated as distinguishable, in which ψ_{n_1} and ψ_{n_2} are the single-particle wavefunctions for the system ($n_1 \neq n_2$). Choose all of the following wavefunctions that are appropriate two-particle stationary state wavefunctions for a system of two non-interacting, identical particles which can be treated as distinguishable.

    (a) $\psi_{n_1}(x_1)\psi_{n_2}(x_1)$ (same label x_1)
    (b) $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$
    (c) $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ (same label n_1)
    (d) $\psi_{n_1}(x)\psi_{n_1}(x)$ (same label x)

Consider the following conversation regarding the appropriate wavefunctions for a system of two non-interacting identical particles that can be treated as **distinguishable**.
Student 1: For a system of two non-interacting identical particles which can be treated as distinguishable, the wavefunction describing the system can be $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ in which $n_1 \neq n_2$. $\psi_{n_1}(x_1)$ means that particle 1 with coordinate x_1 is in a single-particle energy state denoted by n_1. Similarly, $\psi_{n_2}(x_2)$ means that particle 2 with coordinate x_2 is in a single-particle energy state denoted by n_2.
Student 2: I agree with Student 1. Additionally, $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ is also a valid wavefunction for two identical particles which can be treated as distinguishable as there is nothing prohibiting both particles from occupying the same single-particle state with label n_1.
Student 3: Only for the case when both particles occupy the same single-particle state ψ_{n_1} is the two-particle wavefunction $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ the same as for the case of identical bosons.

Explain why you agree or disagree with the students.

Consider the following conversation regarding constructing a wavefunction for a system of N non-interacting identical particles which can be treated as distinguishable from the corresponding single-particle wavefunctions ψ_{n_i}, $i = 1, 2, \ldots, \infty$.

Student 1: For a system of N non-interacting identical particles which can be treated as distinguishable, a stationary state wavefunction describing the system must be a product of the single-particle wavefunctions, i.e., $\Psi(x_1, x_2, \ldots, x_N) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)\cdots\psi_{n_N}(x_N)$.

Student 2: How can the stationary state wavefunction describing the system be the product of the single-particle wavefunctions $\Psi(x_1, x_2, \ldots, x_N) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)\cdots\psi_{n_N}(x_N)$ when the Hamiltonian for a system of the N non-interacting identical particles which can be treated as distinguishable is the sum of the Hamiltonians of each particle $\hat{\mathbf{H}} = \sum_{i=1}^{N} \hat{\mathbf{H}}_i$?

Student 3: Let's consider the stationary state wavefunction to be the product of the single-particle wavefunctions $\Psi(x_1, x_2, \ldots, x_N) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)\cdots\psi_{n_N}(x_N)$. From the TISE, $\hat{\mathbf{H}}\Psi = E\Psi$, where $\hat{\mathbf{H}}$ is the Hamiltonian, Ψ is a stationary state wavefunction, and E is the energy of the many-particle system. Thus,

$$
\begin{aligned}
&\hat{\mathbf{H}}\Psi(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} \hat{\mathbf{H}}_i \Psi(x_1, x_2, \ldots, x_N) \\
&= \sum_{i=1}^{N} \hat{\mathbf{H}}_i \left(\prod_{j=1}^{N} \psi_{n_j}(x_j) \right) \\
&= (\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2 + \ldots + \hat{\mathbf{H}}_N)(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) \\
&= \hat{\mathbf{H}}_1(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) + \hat{\mathbf{H}}_2(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) \\
&+ \ldots + \hat{\mathbf{H}}_N(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) \\
&= E_{n_1}(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) + E_{n_2}(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) \\
&+ \ldots + E_{n_N}(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) \\
&= (E_{n_1} + E_{n_2} + \ldots + E_{n_N})(\psi_{n_1}(x_1)\psi_{n_2}(x_2)\ldots\psi_{n_N}(x_N)) \\
&= (E_{n_1} + E_{n_2} + \ldots + E_{n_N}) \left(\prod_{i=1}^{N} \psi_{n_i}(x_i) \right) \\
&= \left(\sum_{i=1}^{N} E_{n_i} \right) \left(\prod_{j=1}^{N} \psi_{n_j}(x_j) \right) \\
&= E \left(\prod_{i=1}^{N} \psi_{n_j}(x_j) \right) \\
&= E\Psi(x_1, x_2, \ldots, x_N)
\end{aligned}
$$

which is the constant E times the same wavefunction and so $\prod_{i=1}^{N} \psi_{n_i}(x_i)$ is a many-particle stationary state wavefunction. Therefore, the stationary state wavefunction for a system of N non-interacting particles which can be treated as distinguishable is a product of the single-particle wavefunctions.

Explain why you agree or disagree with Student 1 and Student 3 .

25. Write the wavefunction for a system of two **non-interacting**, identical particles which can be treated as distinguishable in which particle 1 is in the single-particle state labeled by n_1 and particle 2 is in a single-particle state labeled by n_2 with $n_1 \neq n_2$. Do not forget to use appropriate coordinates for each particle.

26. Is the wavefunction in question 25 a stationary state wavefunction for a system of two non-interacting identical particles which can be treated as distinguishable? Explain.

27. What is the energy for a system of two **non-interacting** identical particles which can be treated as distinguishable in which particle 1 is in the single-particle state labeled by n_1 and particle 2 is in a single-particle state labeled by n_2?

28. Compare your answer for question 27 to the energy for a system of two indistinguishable particles (questions 10 and 19for fermions and bosons, respectively) where one particle is in a single-particle state labeled by n_1 and the other particle is in a single-particle state labeled by n_2.

29. For a system of N **non-interacting** identical particles which can be treated as distinguishable, write the stationary state wavefunction for the N-particle system, in which ψ_{n_i} is the single-particle wavefunction for the i^{th} particle. Do not forget to use appropriate coordinates for each particle.

30. Write the stationary state wavefunctions for a system of two non-interacting indistinguishable fermions and a system of two indistinguishable bosons (for the distinct single-particle states ψ_{n_1} and ψ_{n_2}) and compare to the stationary state wavefunction for a system of two non-interacting identical particles which can be treated as distinguishable in question 25.

**** Checkpoint: Check your answer to questions 24-30. ****

24. b and c
25. $\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$
26. Yes. $\hat{\mathbf{H}}\Psi = (\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\psi_{n_1}(x_1)\psi_{n_2}(x_2) = (E_{n_1} + E_{n_2})\psi_{n_1}(x_1)\psi_{n_2}(x_2) = E\Psi(x_1, x_2)$
27. $E = E_{n_1} + E_{n_2}$
28. The energy of a system of two identical particles which are indistinguishable fermions or bosons is the same as the energy for a system of two identical particles which can be treated as distinguishable, for which $E = E_{n_1} + E_{n_2}$ for all three cases.
29. $\Psi(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} \psi_{n_i}(x_i) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)\cdots\psi_{n_N}(x_N)$.
30. The stationary state wavefunctions for two non-interacting identical particles occupying the two distinct single-particle states ψ_{n_1} and ψ_{n_2} are given in the following chart

System	Stationary State Wavefunction
Distinguishable Particles	$\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$ or $\Psi(x_1, x_2) = \psi_{n_2}(x_1)\psi_{n_1}(x_2)$
Indistinguishable Fermions	$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$
Indistinguishable Bosons	$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)]$

The wavefunction for a system of indistinguishable particles must reflect symmetrization requirements.

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

Summary of the Properties of the Wavefunction for Distinguishable Particles

- There is no symmetrization requirement for the many-particle stationary state wavefunction for a system of identical particles which can be treated as distinguishable.
- The wavefunction for a system of non-interacting identical particles which can be treated as distinguishable is the product of the single-particle wavefunctions:
  - $\Psi(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N} \psi_{n_i}(x_i)$.

In two to three sentences, summarize the properties of the wavefunction for identical particles (particles of the same type with the same properties). Be sure to describe the properties of indistinguishable fermions, indistinguishable bosons, and identical particles if they could be treated as distinguishable.

Fill in the table below with the properties of an N-particle system consisting of identical particles.

IDENTICAL PARTICLES	
How would you explain to someone why in an N-particle quantum system consisting of identical particles, the particles must be treated as indistinguishable?	
Type of Particle	**Properties**
INDISTINGUISHABLE FERMIONS	What is the constraint on the spin of a fermion?
	Give an example of a physical system consisting of identical fermions in which the fermions must be treated as indistinguishable.
	What is the symmetrization requirement of the N-particle wavefunction (i.e. Completely symmetric, Completely antisymmetric, or No requirement)?
INDISTINGUISHABLE BOSONS	What is the constraint on the spin of a boson?
	Give an example of a physical system consisting of identical bosons in which the bosons must be treated as indistinguishable.
	What is the symmetrization requirement of the N-particle wavefunction (i.e. Completely symmetric, Completely antisymmetric, or No requirement)?
HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES	What is the symmetrization requirement of the N-particle wavefunction (i.e. Completely symmetric, Completely antisymmetric, or No requirement)?

Construct wavefunctions for the following systems of three non-interacting particles with correct normalization. Use the labels n_1, n_2, and n_3 to represent the single-particle stationary state wavefunctions of the system when necessary. If no such wavefunction is permissible, mark the box with an X.

	All 3 particles in the same single-particle state labeled by n_1.	2 particles in the same single-particle state labeled by n_1 1 particle in a different single-particle state labeled by n_2.	All 3 particles in different single-particle states labeled by n_1, n_2, and n_3.
INDISTINGUISHABLE FERMIONS			
INDISTINGUISHABLE BOSONS			
HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES			

**** Check your answers in the preceding tables.****

<table>
<tr><td colspan="2">

IDENTICAL PARTICLES

How would you explain to someone why in an N-particle quantum system consisting of identical particles, the particles must be treated as indistinguishable?

Nature is found to behave in this manner. A system of identical particles consists of N particles in which all the particles are of the same type with the same properties and the particles must be treated as indistinguishable.
</td></tr>
<tr><td>Type of Particle</td><td>Properties</td></tr>
<tr><td rowspan="3">INDISTINGUISHABLE FERMIONS</td><td>What is the constraint on the spin of a fermion?

The N fermions must all be the same half-integer spin particle.</td></tr>
<tr><td>Give an example of a physical system consisting of identical fermions in which the fermions must be treated as indistinguishable.

Electrons in a metal.</td></tr>
<tr><td>What is the symmetrization requirement of the N-particle wavefunction (i.e. Completely symmetric, Completely antisymmetric, or No requirement)?

Completely antisymmetric</td></tr>
<tr><td rowspan="3">INDISTINGUISHABLE BOSONS</td><td>What is the constraint on the spin of a boson?

The N bosons must all be the same integer spin particle.</td></tr>
<tr><td>Give an example of a physical system consisting of identical bosons in which the bosons must be treated as indistinguishable.

He-4 atoms for which there is overlap of the single-particle wavefunctions (i.e., the average separation between atoms is less than the de Broglie wavelength).</td></tr>
<tr><td>What is the symmetrization requirement of the N-particle wavefunction (i.e. Completely symmetric, Completely antisymmetric, or No requirement)?

Completely symmetric</td></tr>
<tr><td>HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES</td><td>What is the symmetrization requirement of the N-particle wavefunction (i.e. Completely symmetric, Completely antisymmetric, or No requirement)?

No Requirement</td></tr>
</table>

	All 3 Particles in the same single-particle state labeled by n_1.	2 particles in the same single-particle state labeled by n_1 1 particle in a different single-particle state labeled by n_2.	All 3 particles in different single-particle states labeled by n_1, n_2, and n_3.
INDISTINGUISHABLE FERMIONS	X	X	$\frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) - \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2)$ $-\psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) + \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1)$ $+\psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2) - \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)]$
INDISTINGUISHABLE BOSONS	$\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$	$\frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$ $+\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)$ $+\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)]$	$\frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2)$ $\psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3) + \psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1)$ $+\psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2) + \psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)]$
HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES	$\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$	$\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$[3]	$\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$[4]

[3] There are two other possibilities: $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)$ and $\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$

[4] There are five other possibilities: $\psi_{n_1}(x_1)\psi_{n_2}(x_3)\psi_{n_3}(x_2)$, $\psi_{n_1}(x_2)\psi_{n_2}(x_1)\psi_{n_3}(x_3)$, $\psi_{n_1}(x_2)\psi_{n_2}(x_3)\psi_{n_3}(x_1)$, $\psi_{n_1}(x_3)\psi_{n_2}(x_1)\psi_{n_3}(x_2)$, and $\psi_{n_1}(x_3)\psi_{n_2}(x_2)\psi_{n_3}(x_1)$

Summary of the Properties of the Wavefunction for Non-Interacting Identical Particles

- Indistinguishable Fermions
  - The basis states used to construct the many-particle stationary state wavefunction for a system of indistinguishable fermions are written in terms of the products of single-particle wavefunctions.
  - The coordinate corresponding to each particle is different in the many-particle stationary state wavefunction.
  - The many-particle wavefunction describing a system of indistinguishable fermions must be completely antisymmetric with respect to exchange of any two particles.
- Indistinguishable Bosons
  - The basis states used to construct the many-particle stationary state wavefunction for a system of N indistinguishable bosons are written in terms of the products of single-particle wavefunctions.
  - The coordinate corresponding to each particle is different in the many-particle stationary state wavefunction.
  - The many-particle wavefunction describing a system of indistinguishable bosons must be completely symmetric with respect to exchange of any two particles.
- Hypothetical Case: Identical Particles if they could be treated as Distinguishable
  - The basis states for the many-particle stationary state wavefunction for a system of identical particles which can be treated as distinguishable can be written in terms of the product of the single-particle wavefunctions.
  - The coordinate corresponding to each particle is different in the many-particle stationary state wavefunction.
  - There is no symmetrization requirement for the many-particle wavefunction for a system of identical particles which can be treated as distinguishable.

4 Examples of Finding Many-Particle Stationary State Wavefunctions and Energies

4.1 One-Dimensional Infinite Square Well (Ignoring spin)

Recall: The single-particle wavefunctions for the infinite square well are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \qquad 0 < x < a \qquad n = 1, 2, 3, \ldots$$

and the single-particle energies are given by

$$E_n = n^2 \left(\frac{\pi^2\hbar^2}{2ma^2}\right) = n^2 E_1.$$

31. Suppose we have two non-interacting particles, both of mass m, in a one-dimensional infinite square well of width a (the well is between $x = 0$ and $x = a$). Find the ground state and first-excited state energies of the many-particle system for the following cases:

    (a) Indistinguishable fermions. (Ignore spin)

    (b) Indistinguishable bosons. (Ignore spin)

    (c) Hypothetical case: Identical particles which can be treated as distinguishable. (Ignore spin)

32. Construct the ground state and first-excited state wavefunctions for two non-interacting particles in that infinite square well for the following cases:

    (a) Indistinguishable fermions. (Ignore spin)

    (b) Indistinguishable bosons. (Ignore spin)

    (c) Hypothetical case: Identical particles which can be treated as distinguishable. (Ignore spin)

Consider the following conversation regarding finding the ground state energy of the many-particle system in a one-dimensional infinite square well of width a (ignore spin).

Student 1: For a system of two non-interacting identical particles, the energy is $E_{n_1,n_2} = E_{n_1} + E_{n_2} = \left(\frac{n_1^2\pi^2\hbar^2}{2ma^2}\right) + \left(\frac{n_2^2\pi^2\hbar^2}{2ma^2}\right) = \left(n_1^2 + n_2^2\right)\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = \left(n_1^2 + n_2^2\right) E_1$.

Student 2: I agree with Student 1. The ground state energy for a system of two identical particles corresponds to the case in which both particles are in the single-particle state labeled by $n_1 = n_2 = 1$. Thus, the ground state energy of the two-particle system is $E_{1,1} = \left(1^2 + 1^2\right) E_1 = 2E_1$

Student 3: I agree with Student 2 only for the cases in which the two particles are indistinguishable bosons or particles which can be treated as distinguishable. In both cases, the particles are permitted to occupy the same lowest single-particle state labeled by $n_1 = n_2 = 1$. However, two indistinguishable fermions cannot occupy the same single-particle state. The ground state energy for a system of two indistinguishable fermions is $E_{1,2} = E_{2,1} = \left(1^2 + 2^2\right) E_1 = 5E_1$.

Explain why you agree or disagree with the students.

Consider the following conversation regarding finding the first-excited state energy of the many-particle system in a one-dimensional infinite square well of width a (ignore spin).

Student 1: For a system of two non-interacting identical particles, the first-excited state energy is $E_{1,2} = \left(1^2 + 2^2\right) E_1 = 5E_1$.

Student 2: I agree with Student 1 only for the cases in which the identical particles are indistinguishable bosons or identical particles which can be treated as distinguishable. The ground state for a system of two indistinguishable fermions corresponds to the case in which one fermion is in the single-particle state labeled by $n_1 = 1$ and the other fermion is in the single-particle state labeled by $n_2 = 2$. The first-excited state energy for a system of two identical fermions corresponds to the case in which one fermion is in the single-particle state labeled by $n_1 = 1$ and the other fermion is in the single-particle state labeled by $n_2 = 3$. Thus, the first-excited state energy for a system of two fermions is $E_{1,3} = \left(1^2 + 3^2\right) E_1 = 10E_1$.

Explain why you agree or disagree with each student.

Consider the following conversation about finding the ground state wavefunction of the many-particle system involving a one-dimensional infinite square well of width a (ignore spin).

Student 1: For a system of two non-interacting identical particles, the ground state wavefunction is $\Psi(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$.

Student 2: I agree with Student 1 only for the cases in which the identical particles are indistinguishable bosons or particles which can be treated as distinguishable since in both cases the particles are permitted to be in the same single-particle state. However, two indistinguishable fermions must be in different single-particle states and the ground state wavefunction for a system of two indistinguishable fermions must be completely antisymmetric.

Student 3: I agree with Student 2. The ground state wavefunction for a system of two indistinguishable fermions is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right]$.

Explain why you agree or disagree with each student.

Consider the following conversation regarding finding the first-excited state wavefunction of the many-particle system in a one-dimensional infinite square well of width a (ignore spin).

Student 1: For a system of two non-interacting identical particles, the first-excited state wavefunction is $\Psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$.

Student 2: I agree with Student 1 only if the particles can be treated as distinguishable.

Student 3: I agree with Student 2. Also, the first-excited state wavefunction for a system of two indistinguishable bosons ignoring spin is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right]$.

Student 2: I agree with Student 3. Furthermore, the first-excited state wavefunction for a system of two indistinguishable fermions ignoring spin is $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2)\right]$.

Explain why you agree or disagree with each student.

****CHECKPOINT: Check your answers to questions 31-32c. ****

31a. Ground state: $E = E_1 + E_2 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2} = \frac{5\pi^2\hbar^2}{2ma^2} = 5E_1$

First excited state: $E = E_1 + E_3 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{9\pi^2\hbar^2}{2ma^2} = \frac{5\pi^2\hbar^2}{ma^2} = 10E_1$

31b. Ground state: $E = E_1 + E_1 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{ma^2} = 2E_1$

First excited state: $E = E_1 + E_2 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2} = \frac{5\pi^2\hbar^2}{2ma^2} = 5E_1$

31c. Ground state: $E = E_1 + E_1 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{ma^2} = 2E_1$

First excited state: $E = E_1 + E_2 = \frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2} = \frac{5\pi^2\hbar^2}{2ma^2} = 5E_1$

32a. Ground state:
$$\begin{aligned}\Psi(x_1,x_2) &= \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)\right] \\ &= \frac{1}{\sqrt{2}}\left[\frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right) - \frac{2}{a}\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{2\pi}{a}x_1\right)\right]\end{aligned}$$

First excited:
$$\begin{aligned}\Psi(x_1,x_2) &= \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_3(x_2) - \psi_1(x_2)\psi_3(x_1)\right] \\ &= \frac{1}{\sqrt{2}}\left[\frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{3\pi}{a}x_2\right) - \frac{2}{a}\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{3\pi}{a}x_1\right)\right]\end{aligned}$$

32b. Ground state:
$$\begin{aligned}\Psi(x_1,x_2) &= \psi_1(x_1)\psi_1(x_2) \\ &= \frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\end{aligned}$$

First excited:
$$\begin{aligned}\Psi(x_1,x_2) &= \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)\right] \\ &= \frac{1}{\sqrt{2}}\left[\frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right) + \frac{2}{a}\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{2\pi}{a}x_1\right)\right]\end{aligned}$$

32c. Ground state:
$$\begin{aligned}\Psi(x_1,x_2) &= \psi_1(x_1)\psi_1(x_2) \\ &= \frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\end{aligned}$$

First excited:
$$\begin{aligned}\Psi(x_1,x_2) &= \psi_1(x_1)\psi_2(x_2) \\ &= \frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\end{aligned}$$
or
$$\begin{aligned}\Psi(x_1,x_2) &= \psi_2(x_1)\psi_1(x_2) \\ &= \frac{2}{a}\sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\end{aligned}$$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

5 Counting the Number of Distinct Many-Particle States

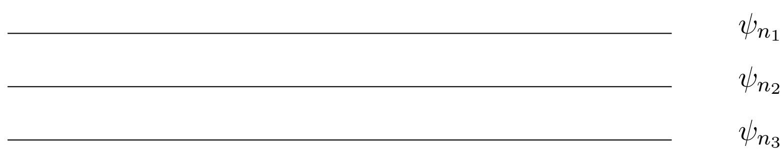
- Now that we know how to construct stationary state wavefunctions from the single-particle wavefunctions for indistinguishable fermions, indistinguishable bosons, and identical particles if they could be treated as distinguishable, let's determine the number of distinct many-particle states for the three different cases, beginning with indistinguishable fermions.
- We will only consider systems in which there is no degeneracy in the single-particle wavefunctions (i.e., $E_{n_i} \neq E_{n_j}$ in which E_{n_i} is the energy corresponding to the single-particle state ψ_{n_i} and E_{n_j} is the energy corresponding to the single-particle state ψ_{n_j})
- Recall: The number of ways to arrange K identical objects among N available slots is $\binom{N}{K} = \frac{N!}{K!(N-K)!}$

CASE I: A Fixed Number of Single Particle States are Available to the System (but the Total Energy of the Many-Particle System is NOT Fixed).

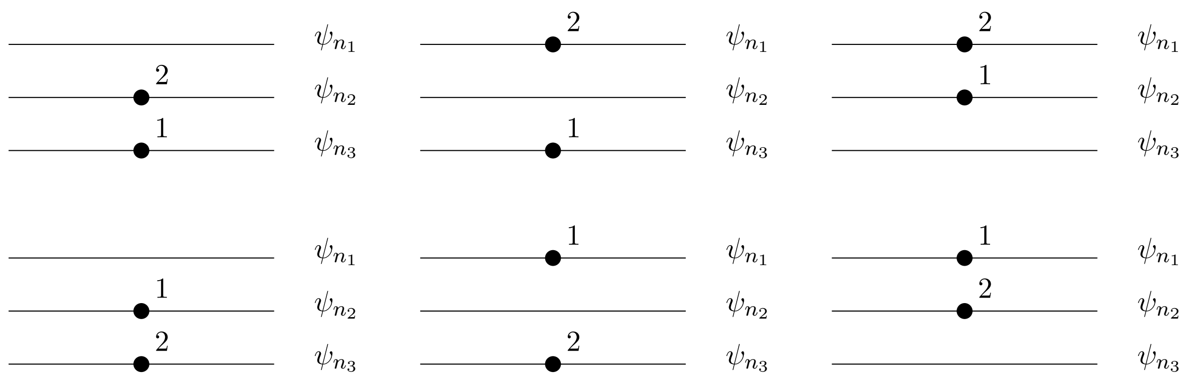
5.1 Determining the Number of Distinct Many-Particle States for **INDISTINGUISHABLE FERMIONS** (no constraints on the total energy of the many-particle system)

5.1.1 Determining the Number of Distinct Many-Particle States for **TWO INDISTINGUISHABLE FERMIONS** and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

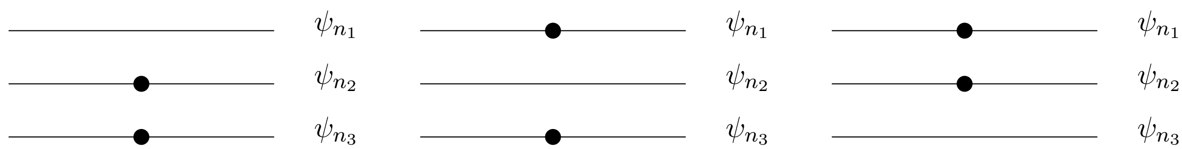
33. Suppose you have two indistinguishable fermions and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct two-particle states can you construct (neglecting spin)? Think about how you could use the diagram below to answer this question by placing the fermions into the single-particle states.



Consider the following conversation regarding the number of distinct two-particle states for a system of two indistinguishable fermions and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}.
Student 1: For a system of two fermions and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}, there are three available single-particle states for the first fermion. That leaves two single-particle states for the second fermion since the second fermion cannot occupy the same single-particle state as the first fermion. The number of two-particle states is $3 \times 2 = 6$.
Student 2: I agree with Student 1. Here is the diagrammatic representation for the 6 distinct two-particle states:



Student 3: I disagree with Student 1 and Student 2. You are overcouting the number of distinct two-particle states. Since the fermions are indistinguishable, we cannot distinguish which fermion is in which single-particle state. We can only tell that one fermion is in single-particle state ψ_{n_2} and another fermion in single-particle state ψ_{n_3}. But there is no way to tell which fermion is in which single-particle state. This indistinguishability is reflected in the antisymmetrized wavefunction. There are 3 distinct two-particle states. Here is the diagrammatic representation for the 3 distinct two-particle states:



Explain why you agree or disagree with each student.

Consider the following conversation regarding the number of distinct two-particle states that you can construct for a system of two indistinguishable fermions and three distinct single-particle states.
Student 1: The Pauli exclusion principle forbids two fermions from occupying the same single-particle state. Each single-particle state can either have one or zero fermions.
Student 2: I agree. There are three distinct single-particle states available to the fermions and we must choose any two for the fermions to occupy. The number of distinct two-particle states for a system of two indistinguishable fermions and three distinct single-particle states is $\binom{3}{2}=\frac{3!}{2!(3-2)!} = 3$.

Explain why you agree or disagree with the students.

5.1.2 Determining the Number of Distinct Many-Particle States for **THREE INDISTINGUISHABLE FERMIONS** and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

34. Suppose you have three indistinguishable fermions and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct three-particle states can you construct (neglecting spin)? If you would like, you can think about how you could use the diagram below to answer this question by placing the fermions into the corresponding states.

ψ_{n_1}

ψ_{n_2}

ψ_{n_3}

Consider the following conversation regarding the number of distinct three-particle states for a system of three indistinguishable fermions.
Student 1: For a system of three indistinguishable fermions and three available single-particle states, there is only one distinct three-particle state. There must be one fermion is each single-particle state.
Student 2: I agree. There are three distinct single-particle states available to the fermions and we must choose three single-particle states for the fermions to occupy. The number of distinct three-particle states for a system of three indistinguishable fermions and three distinct single-particle states is $\binom{3}{3} = \frac{3!}{3!(3-3)!} = 1$.

Explain why you agree or disagree with the students.

5.1.3 Determining the Number of Distinct Many-Particle States for N **INDISTINGUISHABLE FERMIONS** ($N \gg 1$) and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

35. Suppose you have N indistinguishable fermions ($N \gg 1$) and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct N-particle states can you construct (neglecting spin)?

Consider the following conversation regarding the number of distinct three-particle states for a system of N ($N \gg 1$) indistinguishable fermions.

Student 1: For a system of N fermions ($N \gg 1$) and three distinct single-particle states, there is no possible way to place the fermions into the three distinct single-particle states such that no two particles are in the same single-particle state. Therefore, this situation is impossible.

Student 2: I agree. We need at least as many distinct single-particle states available in a situation as the number of fermions in order for such a many-particle system to be possible.

Explain why you agree or disagree with the students.

5.1.4 Determining the Number of Distinct Many-Particle States for N INDISTINGUISHABLE FERMIONS ($N \gg 1$) and M Distinct Single-Particle States ($M \gg 1$) (no constraints on the total energy of the many-particle system)

36. Suppose you have N fermions ($N \gg 1$) and M distinct single-particle states ($M \gg 1$). How many distinct N-particle states can you construct (neglecting spin)?

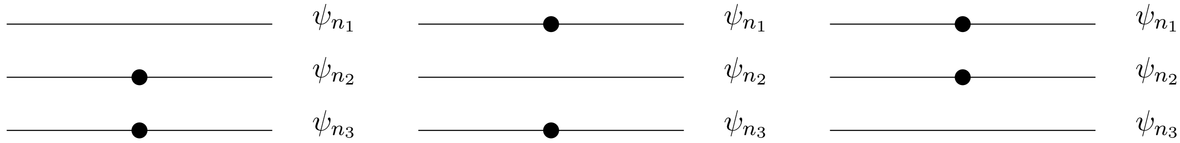
In two to three sentences, describe in words how to determine the number of distinct N-particle states for N indistinguishable fermions and M distinct single-particle states when there are no constraints on the total energy of the many-particle system.

Let's connect the number of distinct single-particle states with the number of possible many-particle stationary state wavefunctions for fermions.

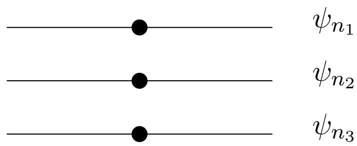
37. Write all the possible two-particle stationary state wavefunctions you found for two indistinguishable fermions in three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3} in question 33.

**CHECKPOINT: Check your answers to questions 33-37. **

33. $\binom{3}{2}$=3



34. $\binom{3}{3}$=1



35. 0. There cannot be more fermions than available single-particle states since that would mean there would be more than one fermion in at least one single-particle state, which is not permitted.

36. The number of distinct N-particle states for a system of N fermions with M available single-particle states is $\begin{cases} \binom{M}{N} & M \geq N \\ 0 & M < N \end{cases}$

37.

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$$

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_3}(x_2) - \psi_{n_3}(x_2)\psi_{n_1}(x_1)]$$

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_2}(x_1)\psi_{n_3}(x_2) - \psi_{n_3}(x_2)\psi_{n_2}(x_1)]$$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

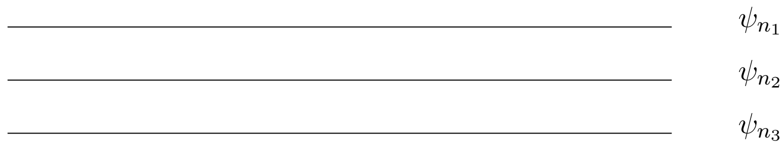
Summary for Determining the Number of Distinct Many-Particle States of INDISTINGUISHABLE FERMIONS for a Fixed Number of Single-Particle States (no constraints on the total energy of the many-particle system)

- The number of distinct N-particle states for a system of N indistinguishable fermions with M available single-particle states when $N \leq M$ is $\binom{M}{N}$.
- The number of distinct N-particle states for a system of N indistinguishable fermions with M available single-particle states when $N > M$ is 0.

5.2 Determining the Number of Distinct Many-Particle States for **INDISTINGUISHABLE BOSONS** (no constraints on the total energy of the many-particle system)

5.2.1 Determining the Number of Distinct Many-Particle States for **TWO INDISTINGUISHABLE BOSONS** and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

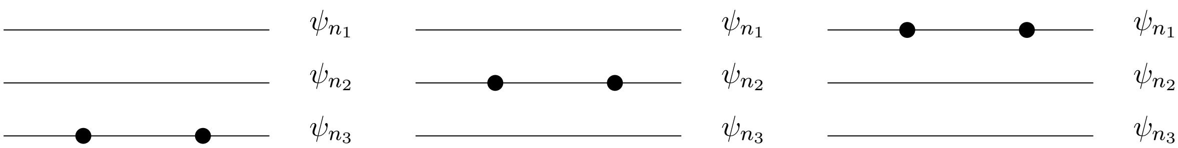
38. Suppose you have two indistinguishable bosons and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct two-particle states can you construct (neglecting spin)? Think about how you could use the diagram below to answer this question by placing the bosons into the corresponding single-particle states.



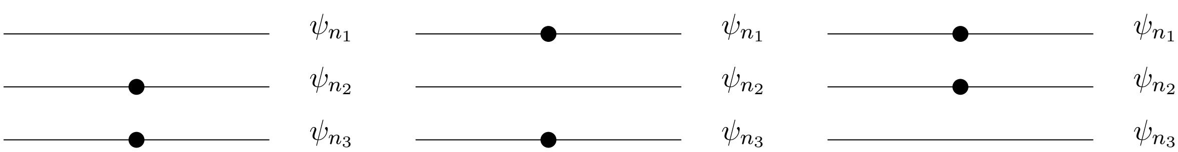
Consider the following conversation regarding the number of distinct two-particle states for a system of two indistinguishable bosons and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3} available.

Student 1: For a system of two bosons and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}, there are three available states for the first boson and three available states for the second boson. The number of two-particle states is $3 \times 3 = 9$.

Student 2: I disagree with Student 1. You are overcounting since you are not taking into account the fact that bosons are indistinguishable. If the bosons are in the same single-particle state, there are three possibilities as follows:



But, if the bosons are in different single-particle states, there are three possibilities since bosons are indistinguishable and swapping the two bosons in the two single-particle states in each of the following situations does not produce a new two-particle state:



There are 6 distinct two-particle states for a system of two bosons and three distinct single-particle states.

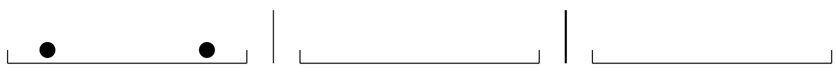
Explain why you agree or disagree with each student.

Consider the following conversation about a method for determining the number of distinct ways two indistinguishable bosons can be arranged in the three distinct single-particle states.

Student 1: For a system of two bosons, there can be more than one boson in a given single-particle state. We can treat the single-particle states as bins to be filled with bosons and dividers to separate the different single-particle states or bins. For example, if the system had two bosons in the first single-particle state then the first bin would have two bosons. For a system with three single-particle states available, we would need two dividers between the three single-particle states. In the case of three single-particle states and two bosons, we must find the number of possible arrangements of the two bosons and two dividers.

Student 2: I agree with Student 1. Furthermore, since the two dividers cannot be distinguished from one another and the bosons cannot be distinguished from one another, we can permute the indistinguishable dividers with the indistinguishable bosons to find all possible ways to permute two bosons in the three single-particle states as follows:

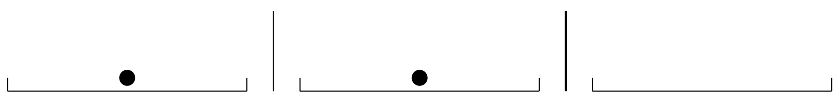
Two Bosons in the First State



Two Bosons in the Second State

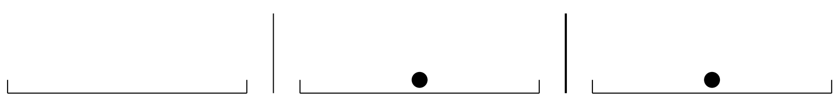
Two Bosons in the Third State

One Boson in the First State and One Boson in the Second State



One Boson in the First State and One Boson in the Third State

One Boson in the Second State and One Boson in the Third State



Student 3: I agree with both Student 1 and Student 2. The number of distinct many-particle states comes from the number of ways the two bosons and two dividers can be permuted. We have a total of four objects (two bosons and two dividers) and we can find the number of ways to permute the two bosons or equivalently the number of ways to permute the two dividers among the four objects. The number of distinct two-particle states is $\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$.

Student 2: Yes! Since the dividers are indistinguishable, permuting them with each other does not give us a new two-particle state. Similarly, since the bosons are indistinguishable, permuting them with each other does not give us a new two-particle state.

Explain why you agree or disagree with the students.

5.2.2 Determining the Number of Distinct Many-Particle States for **THREE INDISTINGUISHABLE BOSONS** and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

39. Suppose you have three indistinguishable bosons and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct three-particle states can you construct (neglecting spin)? If you would like, you can think about how you could use the diagram below to answer this question by placing the bosons into the corresponding states.

ψ_{n_1}

ψ_{n_2}

ψ_{n_3}

Consider the following conversation regarding determining the number of distinct ways three indistinguishable bosons can be arranged in the three distinct single-particle states.

Student 1: Using the bin and divider method, we have three bosons and three bins or single-particle states constructed with two dividers. There are five total objects, three bosons and two dividers, and we must calculate the number of distinct permutations remembering that the bosons are indistinguishable and the dividers are indistinguishable.

Student 2: I agree. We can find the number of ways to permute the three bosons among the five total objects or equivalently the number of ways to permute the two dividers among the five total objects. When we calculate the number of ways to place the two dividers between the three bins, we get $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10$. If instead, we calculate the number of ways to place the three bosons among the two dividers, we get $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = 10$. Either way it is the same!

Explain why you agree or disagree with the students.

5.2.3 Determining the Number of Distinct Many-Particle States for N **INDISTINGUISHABLE BOSONS** ($N \gg 1$) and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

40. Suppose you have N bosons ($N \gg 1$) and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct N-particle states can you construct (neglecting spin)?

Consider the following conversation regarding determining the number of distinct ways N indistinguishable bosons can be arranged in the three distinct single-particle states.

Student 1: Using the bin and divider method, there are $N+2$ total objects to be permuted out of which the N bosons are indistinguishable from each other and the two dividers are indistinguishable from each other. We must calculate the number of distinct arrangements.

Student 2: I agree. When we calculate the number of ways to place the two dividers among the N bosons, we get $\binom{N+2}{2} = \frac{(N+2)!}{2![(N+2)-2)]!} = \frac{(N+2)!}{2!N!} = \frac{(N+2)(N+1)}{2}$. If instead, we calculate the number of ways to place the N bosons among the two dividers, we get $\binom{N+2}{N} = \frac{(N+2)!}{N![(N+2)-N)]!} = \frac{(N+2)!}{N!2!} = \frac{(N+2)(N+1)}{2}$.

Explain why you agree or disagree with each student.

5.2.4 Determining the Number of Distinct Many-Particle States for N INDISTINGUISHABLE BOSONS ($N \gg 1$) and and M Distinct Single-Particle States ($M \gg 1$), (no constraints on the total energy of the many-particle system)

41. Suppose you have N bosons ($N \gg 1$) and M distinct single-particle states ($M \gg 1$). How many distinct N-particle states can you construct (neglecting spin)?

Consider the following conversation regarding determining the number of distinct ways N indistinguishable bosons can be arranged in the M distinct single-particle states.

Student 1: Using the bin and divider method, there are $N+M-1$ total objects that must be permuted, out of which N bosons are indistinguishable from each other and the $M-1$ dividers are indistinguishable from each other. We must calculate the number of distinct arrangements.

Student 2: I agree. When we choose the number of ways to place the $M-1$ indistinguishable dividers among the N bosons, we get $\binom{N+M-1}{M-1} = \frac{(N+M-1)!}{(M-1)![(N+M-1)-(M-1))]!} = \frac{(N+M-1)!}{(M-1)!N!}$. If instead we choose the number of ways to place the N bosons among $M-1$ dividers, we get $\binom{N+M-1}{N} = \frac{(N+M-1)!}{N![(N+M-1)-N)]!} = \frac{(N+M-1)!}{N!(M-1)!}$. Either way it is the same!

Explain why you agree or disagree with the students.

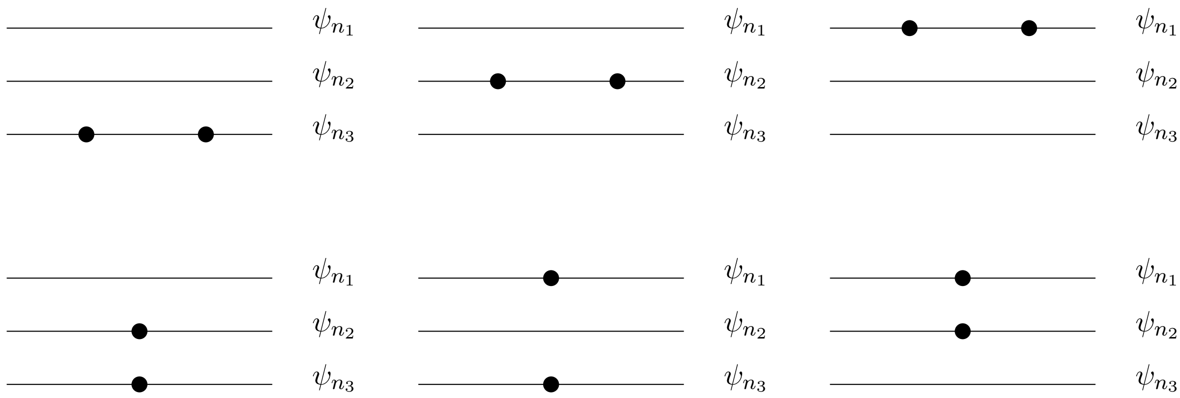
In two to three sentences, describe how to determine the number of distinct N-particle states for N indistinguishable bosons and M distinct one-particle states.

Let's connect the number of distinct many-particle states with the number of possible many-particle stationary state wavefunctions for bosons.

42. Write the two-particle stationary state wavefunctions for the two indistinguishable bosons in three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3} in question 38.

****CHECKPOINT: Check your answers to questions 38-42. ****

38. $\binom{4}{2} = 6$



39. $\binom{5}{2} = 10$

40. $\binom{N+2}{N} = \frac{(N+2)(N+1)}{2}$

41. $\binom{N+M-1}{N} = \binom{N+M-1}{M-1} = \frac{(N+M-1)!}{N!(M-1)!}$

42.

$$\Psi(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_1}(x_2)$$

$$\Psi(x_1, x_2) = \psi_{n_2}(x_1)\psi_{n_2}(x_2)$$

$$\Psi(x_1, x_2) = \psi_{n_3}(x_1)\psi_{n_3}(x_2)$$

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$$

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_3}(x_2) + \psi_{n_1}(x_2)\psi_{n_3}(x_1)]$$

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_2}(x_1)\psi_{n_3}(x_2) + \psi_{n_2}(x_2)\psi_{n_3}(x_1)]$$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

Summary for Determining the Number of Distinct Many-Particle States of INDISTINGUISHABLE BOSONS for a Fixed Number of Single-Particle States (no constraints on the total energy of the many-particle system)

- The number of distinct N-particle states for a system of N indistinguishable bosons with M available single-particle states is
  $\binom{N+M-1}{N} = \binom{N+M-1}{M-1} = \frac{(N+M-1)!}{N!(M-1)!}$

5.3 Hypothetical Case: Determining the Number of Distinct Many-Particle States for **IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE** (no constraints on the total energy of the many-particle system)

- Now that we know how to determine the number of distinct many-particle states for indistinguishable fermions and indistinguishable bosons, let's consider a contrasting case in which the particles can be treated as distinguishable.
- Next, compare the resulting number of many-particle states to what was obtained for indistinguishable fermions and indistinguishable bosons to learn why care must be taken to ensure that the many-particle wavefunction reflects the indistinguishability of the particles.

5.3.1 Determining the Number of Distinct Many-Particle States for **TWO IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE** and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

43. Suppose you have two identical particles which can be treated as distinguishable and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct two-particle states can you construct (neglecting spin)? Think about how you could use the diagram below to answer this question by placing the distinguishable particle into the single-particle states.

ψ_{n_1}

ψ_{n_2}

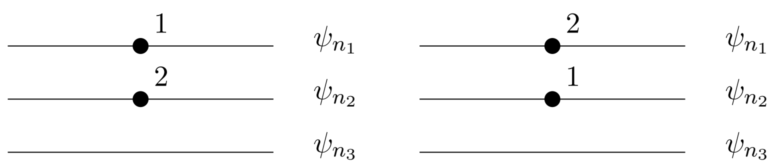
ψ_{n_3}

Consider the following conversation regarding the number of distinct two-particle states for a system of two identical particles which can be treated as distinguishable and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}.

Student 1: The first particle can be placed in one of the three states so there are three possibilities. The same is true about the second particle since there is no restriction on how many particles can be placed in a given single-particle state. Thus, the total number of distinct two-particle states for the system of two identical particles which can be treated as distinguishable with three available single-particle states is $3 \times 3 = 9$.

Student 2: I disagree with Student 1. You are double counting when the particles occupy the same two single-particle states. For example, you are counting the states $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ and $\psi_{n_1}(x_2)\psi_{n_2}(x_1)$ as two distinctly different states. However, there must be only one distinctly different state $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$ in which one particle is in the state labeled by ψ_{n_1} and the other particle is in the state labeled by ψ_{n_2}.

Student 3: I agree with Student 1. There are three two-particle states when the particles are in the same single-particle state and six two-particle states when the particles are in different single-particle states. Since the particles can be treated as distinguishable, we know which particles is in which state. $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$ is the stationary state wavefunction corresponding to particle 1 in the single particle state ψ_{n_1} and particle 2 in the single particle state ψ_{n_2}. Also, particle 1 in state ψ_{n_1} and particle 2 in state ψ_{n_1} is different than particle 2 in state ψ_{n_1} and particle 1 in state ψ_{n_1}. These are two possible stationary state wavefunctions and must be determined as two distinct two-particle states as illustrated in the diagram below.

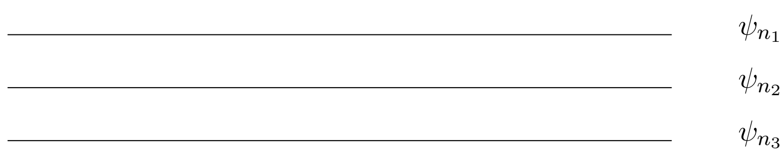


Explain why you agree or disagree with each student.

Student 1 and Student 3 are correct in the previous conversation. Let's extend the rationale to three identical particles which can be treated as distinguishable.

5.3.2 Determining the Number of Distinct Many-Particle States for **THREE IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE** and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

44. Suppose you have three identical particles which can be treated as distinguishable and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct three-particle states can you construct (neglecting spin)? If you would like, think about how you could use the diagram below to answer this question by placing the distinguishable particles into the single-particle states.



Consider the following conversation regarding the number of distinct three-particle states for a system of three identical particles which can be treated as distinguishable and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}.

Student 1: The first particle can be placed in one of the three states so there are three possibilities. The same is true for the second particle and the third particle since there is no restriction on how many particles we can place in a given single-particle state. The total number of distinct three-particle states for the system of three identical particles which can be treated as distinguishable with three available single-particle states is $3 \times 3 \times 3 = 27$.

Student 2: I agree with Student 1. The total number of distinct three-particle states for the system of three identical particles which can be treated as distinguishable with three available single-particle states is

$$[\text{Three single-particle states}]^{(\text{Three Particles})} = 3^3 = 27.$$

Student 3: I agree with both Student 1 and Student 2. And in general, the total number of distinct states for a system of identical particles which can be treated as distinguishable is

$$[\text{Number of Single-Particle States}]^{(\text{Number of Particles})}.$$

Explain why you agree or disagree with the students.

5.3.3 Determining the Number of Distinct Many-Particle States for N **IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE** ($N \gg 1$) and Three Distinct Single-Particle States (no constraints on the total energy of the many-particle system)

45. Suppose you have N identical particles which can be treated as distinguishable ($N \gg 1$) and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. How many distinct N-particle states can you construct (neglecting spin)?

5.3.4 Determining the Number of Distinct Many-Particle States for N **IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE** ($N \gg 1$) and M Distinct Single-Particle States ($M \gg 1$) (no constraints on the total energy of the many-particle system)

46. Suppose you have N identical particles which can be treated as distinguishable ($N \gg 1$) and M distinct single-particle states ($M \gg 1$). How many distinct N-particle states can you construct?

In two to three sentences, summarize how to determine the number of distinct N-particle states for N identical particles which can be treated as distinguishable and M distinct single-particle states.

Rank the number of distinct N-particle states for identical particles if they are indistinguishable fermions, indistinguishable bosons, or identical particles that can be treated as distinguishable for N identical particles ($N \gg 1$) and M distinct single-particle states ($M \gg 1$).

Let's connect the number of distinct single-particle states with the number of possible stationary state wavefunctions for identical particles which can be treated as distinguishable.

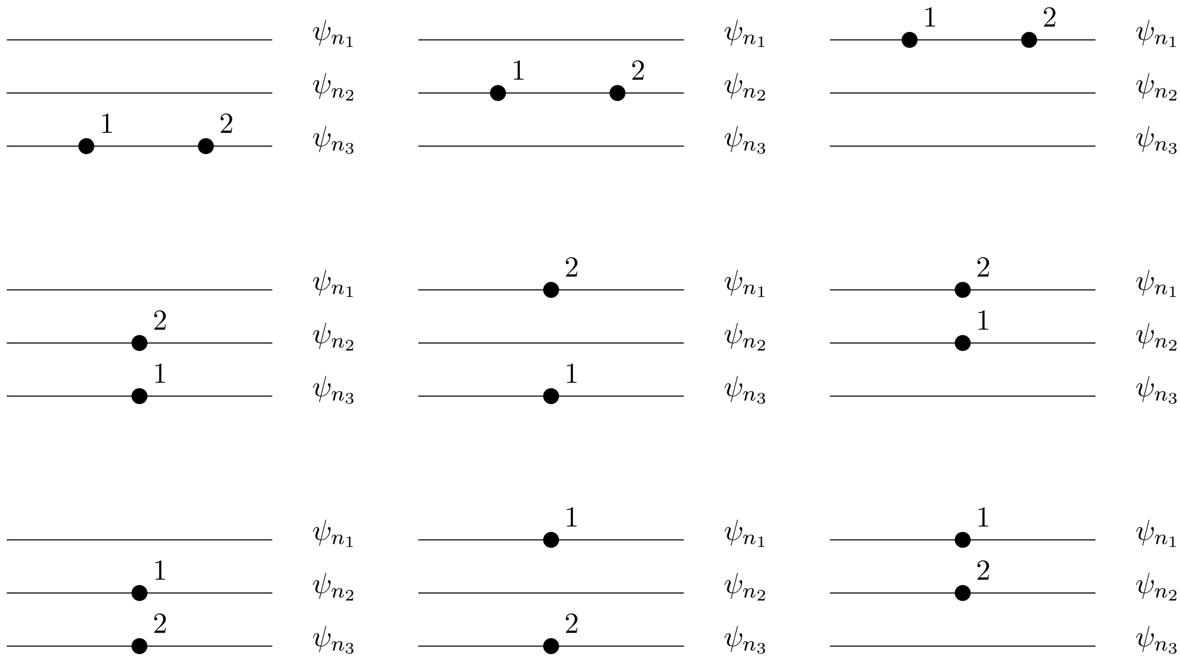
47. Write all of the possible two-particle stationary state wavefunctions you found for two identical particles which can be treated as distinguishable in three distinct single-particle states given by the wavefunctions ψ_{n_1}, ψ_{n_2}, and ψ_{n_3} in question 43 for the following situations:

- Both particles are in the same single-particle state: (Hint: There are three possible two-particle stationary state wavefunctions).

- Two particles are in different single-particle states: (Hint: There are six possible two-particle stationary state wavefunctions).

****CHECKPOINT: Check your answers to questions 43-46. ****

43. $3 \times 3 = 3^2 = 9$



44. $3 \times 3 \times 3 = 3^3 = 27$
45. 3^N
46. M^N
47.

Two particles in the same state:
$$\begin{aligned}
\Psi(x_1, x_2) &= \psi_{n_1}(x_1)\psi_{n_1}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_2}(x_1)\psi_{n_2}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_3}(x_1)\psi_{n_3}(x_2)
\end{aligned}$$

Two particles in different states:
$$\begin{aligned}
\Psi(x_1, x_2) &= \psi_{n_1}(x_1)\psi_{n_2}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_1}(x_1)\psi_{n_3}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_2}(x_1)\psi_{n_1}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_2}(x_1)\psi_{n_3}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_3}(x_1)\psi_{n_1}(x_2) \\
\Psi(x_1, x_2) &= \psi_{n_3}(x_1)\psi_{n_2}(x_2)
\end{aligned}$$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

Summary for Determining the Number of Distinct Many-Particle States of IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE for a Fixed Number of Single-Particle States (no constraints on the total energy of the many-particle system)

- The number of distinct N-particle states for a system of N identical particles if they could be treated as distinguishable with M available single-particle states is M^N.

To summarize what you have learned about determining the number of distinct many-particle states for a fixed number of single-particle states (total energy of the many-particle system is not fixed), fill in the following table with how many disinct many-particle states you can construct for the given situation.

Identical Particles	
INDISTINGUISHABLE FERMIONS	5 particles and 7 distinct single-particle states
	5 particles and 3 distinct single-particle states
INDISTINGUISHABLE BOSONS	5 particles and 7 distinct single-particle states
	5 particles and 3 distinct single-particle states
HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES	5 particles and 7 distinct single-particle states
	5 particles and 3 distinct single-particle states

In two to three sentences, summarize how to determine the number of distinct N-particle states for N identical particles and M distinct single-particle states. Be sure to describe the cases of indistinguishable fermions, indistinguishable bosons, and the hypothetical case of identical particles which can be treated as distinguishable.

Review your answers to the questions in the preceding table for the given system of identical particles for a fixed number of single-particle states (no constraints on the total energy of the many-particle system).

Identical Particles	
INDISTINGUISHABLE FERMIONS	5 particles and 7 distinct single-particle states $\binom{7}{5} = 21$
	5 particles and 3 distinct single-particle states None, there are more particles than available states.
INDISTINGUISHABLE BOSONS	5 particles and 7 distinct single-particle states $\binom{11}{5}=\binom{11}{6}=462$
	5 particles and 3 distinct single-particle states $\binom{7}{5}=\binom{7}{2}=21$
HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES	5 particles and 7 distinct single-particle states $7^5 = 16,807$
	5 particles and 3 distinct single-particle states $3^5 = 243$

Summary of CASE I: Determining the Number of Distinct Many-Particle States for a Fixed Number of Single-Particle States (no constraints on the total energy of the many-particle system)

- Indistinguishable Fermions
  - The number of distinct N-particle states for a system of N indistinguishable fermions with M available single-particle states when $N \leq M$ is $\binom{M}{N}$.
  - The number of distinct N-particle states for a system of N indistinguishable fermions with M available single-particle states when $N > M$ is 0 (such a state is NOT possible).
- Indistinguishable Bosons
  - The number of distinct N-particle states for a system of N indistinguishable bosons with M available single-particle states is $\binom{N+M-1}{N} = \binom{N+M-1}{M-1} = \frac{(N+M-1)!}{N!(M-1)!}$
- Identical Particles which are Distinguishable
  - The number of distinct N-particle states for a system of N identical particles which can be treated as distinguishable with M available single-particle states is M^N.

CASE II: Determining the Number of Distinct Many-Particle States when the Total Energy of the Many-Particle System is Fixed (Ignore spin).

- Let's consider three non-interacting identical particles of mass m in a one-dimensional infinite square well of width "a".
- Recall that the total energy of the many-particle system can be written in terms of the single-particle energies as

$$E = E_{n_1} + E_{n_2} + E_{n_3} = (n_1^2 + n_2^2 + n_3^2)\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = (n_1^2 + n_2^2 + n_3^2)E_1.$$

  Here n_1, n_2, n_3 are positive integers that label the single-particle states in which the three particles can be placed.
- Suppose the total energy is $E = 243\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = 243E_1$
- Note: The only sets of integers n_1, n_2, and n_3 whose squares sum to 243 are given below:

$$\begin{aligned} 243 &= 1^2 + 11^2 + 11^2 \\ 243 &= 3^2 + 3^2 + 15^2 \\ 243 &= 5^2 + 7^2 + 13^2 \\ 243 &= 9^2 + 9^2 + 9^2. \end{aligned}$$

48. List all of the combinations of three positive integers (n_1, n_2, n_3) whose squares sum to 243. For example, two combinations would be $(1, 11, 11)$ and $(11, 1, 11)$.

5.4 Determining the Number of Distinct Many-Particle States for Three **INDISTINGUISHABLE FERMIONS** in a One-Dimensional Infinite Square Well with a Fixed Total Energy for the Many-Particle System

49. Suppose you have three indistinguishable fermions and the total energy of the three-particle system is $E = 243\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = 243E_1$. How many distinct three-particle states can you construct? [Hint: Consider the combinations in question 48 that are possible for indistinguishable fermions and the antisymmetric requirement for the wavefunction.]

50. Write all of the possible three-particle stationary state wavefunctions for the system of three indistinguishable fermions in the one-dimensional infinite square well with total energy $E = 243E_1$. (The Slater determinant may be helpful.)

Consider the following conversation regarding the number of distinct three-particle states you can construct for a system of three indistinguishable fermions with a total energy of $E = 243E_1$.

Student 1: For a system of three indistinguishable fermions with a total energy of $E = 243E_1$, there is only one three-particle state. There is one fermion in the single-particle state ψ_5, one fermion in the state ψ_7, and one fermion in the state ψ_{13}.

Student 2: I disagree with Student 1. There are four disinct three-particle states for the three fermions: $\psi_1(x_1)\psi_{11}(x_2)\psi_{11}(x_3)$, $\psi_3(x_1)\psi_3(x_2)\psi_{15}(x_3)$, $\psi_5(x_1)\psi_7(x_2)\psi_{13}(x_3)$, and $\psi_9(x_1)\psi_9(x_2)\psi_9(x_3)$.

Student 3: I agree with Student 1. There cannot be more than one fermion in each single-particle state. The combination $(9, 9, 9)$ is a system with three fermions in the state ψ_9. The combinations $(3, 3, 15)$, $(3, 15, 3)$, and $(15, 3, 3)$ have two fermions in the state ψ_3 and the combinations $(1, 11, 11)$, $(11, 1, 11)$, and $(11, 11, 1)$ have two fermions in the state ψ_{11}. None of these are possible for fermions.

Student 1: I agree with Student 3. A system of indistinguishable fermions must satisfy the antisymmetrization requirement, so there is only one distinct three-particle state, corresponding to the combinations $(5, 7, 13), (5, 13, 7), (7, 5, 13), (7, 13, 5), (13, 5, 7)$, and $(13, 7, 5)$.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the number of three-particle states you can construct for a system of three indistinguishable fermions with total energy of $E = (5^2 + 7^2 + 13^2)\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = 243E_1$.

Student 1: How can there only be one distinct three-particle state for a system of three indistinguishable fermions corresponding to the six combinations $(5, 7, 13), (5, 13, 7), (7, 5, 13), (7, 13, 5), (13, 5, 7)$, and $(13, 7, 5)$?

Student 2: Since the fermions are indistinguishable, we cannot say which fermion is in which single-particle state. All we can say is that one fermion is in the single-particle state ψ_5, one fermion is in the single-particle state ψ_7, and one fermion is in the single-particle state ψ_{13}. The stationary state wavefuntion for the three indistinguishable fermions must be completely antisymmetric. The six combinations $(5, 7, 13), (5, 13, 7), (7, 5, 13), (7, 13, 5), (13, 5, 7)$, and $(13, 7, 5)$ correspond to the labels for the products of the single-particle states to be summed to obtain the three-particle stationary state wavefunction.

Student 3: I agree with Student 2. To find the three-particle stationary state wavefunction for a system of three indistinguishable fermions, we must ensure that the wavefunction is completely antisymmetric and normailized. The normalization factor is $\frac{1}{\sqrt{3!}}$. We can use the Slater determinant to ensure that we include all the terms with the correct sign and obtain

$$\frac{1}{\sqrt{6}}\begin{vmatrix} \psi_5(x_1) & \psi_7(x_1) & \psi_{13}(x_1) \\ \psi_5(x_2) & \psi_7(x_2) & \psi_{13}(x_2) \\ \psi_5(x_3) & \psi_7(x_3) & \psi_{13}(x_3) \end{vmatrix} = \begin{array}{l} \frac{1}{\sqrt{6}}[\psi_5(x_1)\psi_7(x_2)\psi_{13}(x_3) - \psi_5(x_1)\psi_{13}(x_2)\psi_7(x_3) \\ -\psi_7(x_1)\psi_5(x_2)\psi_{13}(x_3) + \psi_7(x_1)\psi_{13}(x_2)\psi_5(x_3) \\ +\psi_{13}(x_1)\psi_5(x_2)\psi_7(x_3) - \psi_{13}(x_1)\psi_7(x_2)\psi_5(x_3)]. \end{array}$$

Explain why you agree or disagree with Student 2 and Student 3.

****CHECKPOINT: Check your answers to questions 48-50. ****

48.

$$(9,9,9)$$
$$(3,3,15),(3,15,3),(15,3,3)$$
$$(1,11,11),(11,1,11),(11,11,1)$$
$$(5,7,13),(5,13,7),(7,5,13),(7,13,5),(13,5,7),(13,7,5)$$

49. 1. Two or more fermions in the same single-particle state are not possible. Identical fermions must satisfy the antisymmetrization requirement.

50.

$$\Psi(x_1,x_2,x_3) = \frac{1}{\sqrt{6}}[\psi_5(x_1)\psi_7(x_2)\psi_{13}(x_3) - \psi_5(x_1)\psi_{13}(x_2)\psi_7(x_3) - \psi_7(x_1)\psi_5(x_2)\psi_{13}(x_3)$$
$$+\psi_7(x_1)\psi_{13}(x_2)\psi_5(x_3) + \psi_{13}(x_1)\psi_5(x_2)\psi_7(x_3) - \psi_{13}(x_1)\psi_7(x_2)\psi_5(x_3)]$$

If your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.

5.5 Determining the Number of Distinct Many-Particle States for Three **INDISTINGUISHABLE BOSONS** in a One-Dimensional Infinite Square Well with a Fixed Total Energy for the Many-Particle System (Ignore Spin)

51. Suppose you have three indistinguishable bosons and the total energy of the three-particle system is $E = 243E_1$. How many distinct three-particle states can you construct? [Hint: Consider the combinations in question 48 that are possible for indistinguishable bosons.]

52. Write all of the possible three-particle stationary state wavefunctions for the system of three indistinguishable bosons in the one-dimensional infinite square well with total energy $E = 243E_1$.

Consider the following conversation regarding the number of three-particle states you can construct for a system of three indistinguishable bosons with total energy $E = 243E_1$.

Student 1: For a system of three indistinguishable bosons with a total energy of $E = 243E_1$, there is only one three-particle state. There is one boson in the state ψ_5, one boson in the state ψ_7, and one boson in the state ψ_{13}.

Student 2: I disagree with Student 1. It is possible for bosons to occupy the same single-particle state. Since the bosons are indistinguishable, there are four disinct three-particle states for the three bosons with the total energy E.

Student 3: I agree with Student 2. All three bosons could be in the state ψ_9. There could also be two bosons in state ψ_3 and one boson in state ψ_{15}, two bosons in state ψ_{11} and one boson in state ψ_1, or one boson in each of the states ψ_5, ψ_7, and ψ_{13}.

Explain why you agree or disagree with each student.

****CHECKPOINT: Check your answers to questions 51-52. ****

51. 4.
52.

$$\Psi(x_1, x_2, x_3) = \psi_9(x_1)\psi_9(x_2)\psi_9(x_3)$$

$$\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{3}}[\psi_3(x_1)\psi_3(x_2)\psi_{15}(x_3) + \psi_3(x_1)\psi_{15}(x_2)\psi_3(x_3) + \psi_{15}(x_1)\psi_3(x_2)\psi_3(x_3)]$$

$$\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{3}}[\psi_1(x_1)\psi_{11}(x_2)\psi_{11}(x_3) + \psi_{11}(x_1)\psi_1(x_2)\psi_{11}(x_3) + \psi_{11}(x_1)\psi_{11}(x_2)\psi_1(x_3)]$$

$$\begin{aligned}\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}[&\psi_5(x_1)\psi_7(x_2)\psi_{13}(x_3) + \psi_5(x_1)\psi_{13}(x_2)\psi_7(x_3) + \psi_7(x_1)\psi_5(x_2)\psi_{13}(x_3) \\ &+\psi_7(x_1)\psi_{13}(x_2)\psi_5(x_3) + \psi_{13}(x_1)\psi_5(x_2)\psi_7(x_3) + \psi_{13}(x_1)\psi_7(x_2)\psi_5(x_3)]\end{aligned}$$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.[5]

[5]Note, the four states can be regarded as a basis for the three-particle system and any linear superposition of the four states listed in question 52 would also be a three-particle stationary state wavefunction for the system of three indistinguishable bosons due to the degeneracy in the energy spectrum. However, in this tutorial we will not focus on the linear superpostion of these states.

5.6 Hypothetical Case: Determining the Number of Distinct Many-Particle States for Three **IDENTICAL PARTICLES IF THEY COULD BE TREATED AS DISTINGUISHABLE** in a One-Dimensional Infinite Square Well with a Fixed Total Energy for the Many-Particle System (Ignore spin)

53. Suppose you have three identical particles which can be treated as distinguishable and the total energy of the three-particle system $E = 243E_1$. How many distinct three-particle states can you construct if the total energy of the many-particle system is fixed? [Hint: Consider the combinations in question 48 that are possible for identical particles which can be treated as distinguishable.]

54. Write four possible three-particle stationary state wavefunctions for a system of three identical particles which can be treated as distinguishable in the one-dimensional infinite square well with total energy $E = 243E_1$.

Consider the following conversation regarding the number of three-particle states you can construct with a total energy $E = 243E_1$ for a system of three identical particles which can be treated as distinguishable.
Student 1: For a system of three identical particles which can be treated as distinguishable with a total energy $E = 243E_1$, there are four distinct three-particle states with wavefunctions: $\psi_1(x_1)\psi_{11}(x_2)\psi_{11}(x_3)$, $\psi_3(x_1)\psi_3(x_2)\psi_{15}(x_3)$, $\psi_5(x_1)\psi_7(x_2)\psi_{13}(x_3)$, and $\psi_9(x_1)\psi_9(x_2)\psi_9(x_3)$.
Student 2: I disagree with Student 1. Since the particles can be treated as distinguishable, we can tell which particle is in which single-particle state. For example, there are three distinct many-particle states corresponding to the particles in the single-particle states ψ_3, ψ_3, and ψ_{15}: $\psi_3(x_1)\psi_3(x_2)\psi_{15}(x_3)$, $\psi_3(x_1)\psi_{15}(x_2)\psi_3(x_3)$, and $\psi_{15}(x_1)\psi_3(x_2)\psi_3(x_3)$. Similarly, there are three distinct states corresponding to the particles in the single-particle states ψ_1,ψ_{11}, and ψ_{11}.
Student 3: I agree with Student 2. There is one distinct many-particle state corresponding to all three particles in the single-particle state, ψ_9 and six distinct many-particle states corresponding to the particles in the single-particle states ψ_5, ψ_7, and ψ_{13} because the particles can be treated as distinguishable.
Student 2: I agree with Student 3. There are 13 distinct many-particle states for the system of three identical particles which can be treated as distinguishable with energy $E = 243E_1$.

Explain why you agree or disagree with each student.

****CHECKPOINT: Check your answers to questions 53-54. ****

53. 13. There are 13 combinations that are distinct for identical particles which can be treated as distinguishable.

54.

$$\Psi(x_1, x_2, x_3) = \psi_9(x_1)\psi_9(x_2)\psi_9(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_3(x_1)\psi_3(x_2)\psi_{15}(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_3(x_1)\psi_{15}(x_2)\psi_3(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_{15}(x_1)\psi_3(x_2)\psi_3(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_1(x_1)\psi_{11}(x_2)\psi_{11}(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_{11}(x_1)\psi_1(x_2)\psi_{11}(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_{11}(x_1)\psi_{11}(x_2)\psi_1(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_5(x_1)\psi_7(x_2)\psi_{13}(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_5(x_1)\psi_{13}(x_2)\psi_7(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_7(x_1)\psi_5(x_2)\psi_{13}(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_7(x_1)\psi_{13}(x_2)\psi_5(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_{13}(x_1)\psi_5(x_2)\psi_7(x_3)$$
$$\Psi(x_1, x_2, x_3) = \psi_{13}(x_1)\psi_7(x_2)\psi_5(x_3)$$

If your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answers.[6]

In two or three sentences, compare the hypothetical case if particles could be treated as distinguishable to the case of indistinguishable fermions and bosons.

[6] Note, the thirteen states can be regarded as a basis for the three-particle system and any linear superposition of the thirteen states listed in question 54 is a three-particle stationary state wavefunction for the system of three distinguishable particles due to the degeneracy in the energy spectrum. However, in this tutorial we will not focus on linear superpostions of these states.

Summary of CASE II: Determining the Number of Distinct Many-Particle States for a Many-Particle System with Fixed Energy (Ignore spin)

To summarize what you have learned about determining the number of distinct many-particle states for a many-particle system with fixed energy, answer the following questions in the table below for a system of two particles in a one-dimensional infinite square well with fixed total energy $E = 200\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = 200E_1$.

(a) What are the possible combinations (i.e., what are the possible combinations of (n_1, n_2) that yield a total energy of $200E_1$ for the two-particle system)?

(b) How many disinct two-particle states can you construct?

Note: The only possible integers n_1 and n_2 whose squares sum to 200 are given below

$$\begin{aligned} 200 &= 10^2 + 10^2 \\ 200 &= 2^2 + 14^2 \end{aligned}$$

Identical Particles	
INDISTINGUISHABLE FERMIONS	(a) Possible combinations (n_1, n_2)
	(b) How many distinct two-particle states?
INDISTINGUISHABLE BOSONS	(a) Possible combinations (n_1, n_2)
	(b) How many distinct two-particle states?
HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES	(a) Possible combinations (n_1, n_2)
	(b) How many distinct two-particle states?

Summary of CASE II: Determining the Number of Distinct Many-Particle States for a Many-Particle System with Fixed Energy (Ignore Spin)

Check your answers to the questions in the preceding table.

Identical Particles	
INDISTINGUISHABLE FERMIONS	(a) Possible combinations (n_1, n_2) $(2, 14), (14, 2)$
	(b) How many distinct two-particle states? 1
INDISTINGUISHABLE BOSONS	(a) Possible combinations (n_1, n_2) $(2, 14), (14, 2), (10, 10)$
	(b) How many distinct two-particle states? 2

HYPOTHETICAL CASE: DISTINGUISHABLE PARTICLES	(a) Possible combinations (n_1, n_2) $(2, 14), (14, 2), (10, 10)$
	(b) How many distinct two-particle states? 3

55. Suppose that for a system of two non-interacting identical particles in a one-dimensional infinite square well, the total energy of the two-particle system is $E_{n_1,n_2} = (n_1^2 + n_2^2)E_1$, in which E_1 is the ground state energy for the single-particle system. The total energy of the two-particle system is $E = 50E_1$. Assume all of the possible combinations are equally probable.[7]
Note: The only pairs of integers n_1 and n_2 whose squares sum to 50 are given below:

$$\begin{aligned} 50 &= 1^2 \quad +7^2 \\ &= 5^2 \quad +5^2. \end{aligned}$$

(a) If the particles are indistinguishable fermions and you randomly measure the energy of one particle, what single-particle energies might you obtain and with what probability? Explain.

(b) If the particles are indistinguishable bosons and you randomly measure the energy of one particle, what single-particle energies might you obtain and with what probability? Explain.

(c) Hypothetical case: If the particles could be treated as distinguishable and you randomly measure the energy of one particle, what single-particle energies might you obtain and with what probability? Explain.

Briefly describe how the probability of the possible values of energy differs in the case of indistinguishable fermions, indistinguishable bosons, and the hypothetical case in which particles can be treated as distinguishable.

[7]Due to the degeneracy of the two-particle system, any linear combination of degenerate two-particle stationary states is a two-particle stationary state with the same energy. However, in this tutorial we will not focus on linear superpostions of these states.

Consider the following conversation regarding the possible outcomes if you measure the energy of a single particle and the corresponding probability if the particles are indistinguishable fermions.

Student 1: For a system of two indistinguishable fermions in which the total energy of the two-particle system is $E = 50E_1$, there are two possible combinations: $(1,7)$ and $(7,1)$. The two combinations contribute to the completely antisymmetric wavefunction in which one fermion is in the state ψ_1 and one fermion is in the state ψ_7.

Student 2: I agree with Student 1. Additionally, the fermions could have the combination $(5,5)$ in which both fermions are in the single-particle state ψ_5. Therefore, if you randomly measure the energy you could obtain the energies E_1, $49E_1$, or $25E_1$ with equal probability 1/3.

Student 1: I disagree with Student 2. The fermions cannot be in the same single-particle state ψ_5. One fermion must be in the single-particle state ψ_1 and one fermion must be in the single-particle state ψ_7. If you randomly measure the energy, you could obtain the energy E_1 or $49E_1$ with equal probability of 1/2.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the possible outcomes if you measure the energy of a single particle and the corresponding probability if the particles are indistinguishable bosons.

Student 1: For a system of two indistinguishable bosons in which the total energy of the two-particle system is $E = 50E_1$, there are three possible combinations: $(1,7)$, $(7,1)$, and $(5,5)$. The combinations $(1,7)$ and $(7,1)$ correspond to the completely symmetric state $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_7(x_2) + \psi_7(x_1)\psi_1(x_2)]$. The combination $(5,5)$ corresponds to two bosons in the same state ψ_5.

Student 2: I agree with Student 1. It is equally probable that the bosons are in the same state ψ_5 or one boson is in the state ψ_1 and the other boson is in the state ψ_7. If you randomly measure the energy you could obtain the energies E_1, $49E_1$, or $25E_1$.

Student 3: I agree with Student 2. Since the three combinations are equally likely, the probability that the system has the combination $(1,7)$, $(7,1)$, or $(5,5)$ is 1/3. For the combination $(1,7)$, the probability of obtaining 1^2E_1 is 1/2. Similarly, the probability of obtaining E_1 for the combination $(7,1)$ is 1/2. Therefore, the probability of obtaining E_1 is $(1/3)\times(1/2)+(1/3)\times(1/2) = 1/3$. By the same reasoning, the probability of obtaining $49E_1$ is $2\times(1/3)\times(1/2) = 1/3$. The probability of the system being in the combination $(5,5)$ is 1/3. For bosons with the combination $(5,5)$, the probability of being in state ψ_5 is 1. Thus, the probability of obtaining $25E_1$ is $(1/3)\times 1 = 1/3$.

Student 1: I agree with Student 2, but disagree with Student 3. The probability of the bosonic system having the combination $(5,5)$ is 1/2 and the probability of having the combinations $(1,7)$ and $(7,1)$, which correspond to one two-particle state $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_7(x_2) + \psi_7(x_1)\psi_1(x_2)]$ is 1/2. The probability of obtaining E_1 is $(1/2)\times(1/2) = 1/4$, the probability of obtaining $49E_1$ is $(1/2)\times(1/2) = 1/4$, and the probability of obtaining $25E_1$ is $(1/2)\times 1 = 1/2$.

Explain why you agree or disagree with each student.

Hypothetical Case: Treating the identical particles as distinguishable.

Consider the following conversation regarding the possible outcomes if you measure the energy of a single particle and the corresponding probability if identical particles could be treated as distinguishable.
Student 1: For a system of two identical particles if they could be treated as distinguishable, there are three possible combinations $(1,7)$, $(7,1)$ and $(5,5)$ if the total energy of the two-particle system is $E = 50E_1$. Each combination is equally probable with probability $1/3$.
Student 2: I agree with Student 1. If identical particles which can be treated as distinguishable are in the combination $(1,7)$ and you measure the energy, you could obtain the energy E_1 with probability $(1/3) \times (1/2) = 1/6$ and the energy $49E_1$ with probability $(1/3) \times (1/2) = 1/6$.
Student 3: I agree with Student 1 and Student 2. If identical particles which can be treated as distinguishable are in the combination $(7,1)$ and you measure the energy, you could obtain the energy E_1 with probability $(1/3) \times (1/2) = 1/6$ and the energy $49E_1$ with probability $(1/3) \times (1/2) = 1/6$.
Student 1: I agree with Student 2 and Student 3. If identical particles which can be treated as distinguishable are in the combination $(5,5)$ and you measure the energy you would obtain the energy, $25E_1$ with probability $(1/3) \times 1 = 1/3$.
Student 2: To sum up, if you randomly measure the energy, you could obtain the energy E_1 with probability $(1/3)\times(1/2)+(1/3)\times(1/2) = 1/3$, the energy $49E_1$ with probability $(1/3)\times(1/2)+(1/3)\times(1/2) = 1/3$, and the energy $25E_1$ with probability $(1/3) \times 1 = 1/3$.

Explain why you agree or disagree with each student.

****CHECKPOINT: Check your answers to questions 55a-55. ****

55a. E_1 with probability $\frac{1}{2}$ or $49E_1$ with probability $\frac{1}{2}$

55b. E_1 with probability $\frac{1}{4}$, $49E_1$ with probability $\frac{1}{4}$ or $25E_1$ with probability $\frac{1}{2}$

55c. E_1 with probability $\frac{1}{3}$, $49E_1$ with probability $\frac{1}{3}$ or $25E_1$ with probability $\frac{1}{3}$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

6 Determining the Number of Distinct Many-Particle States when the Total Energy of the Many-Particle System is Fixed and the Single-Particle States Have Degeneracy

- Here, we will consider a system of identical particles in which there is degeneracy in the single-particle energy spectrum and there are constraints on the number of particles in different single-particle states with a certain energy. We will focus on the spatial part of the wavefunction and ignore the spin degrees of freedom.
- We will consider a group of degenerate states together and the arrangement $(N_1, N_2, N_3, \ldots, N_n, \ldots)$ is such that for all of the single-particle states with energy E_i, the total number of particles is N_i. We will use the notation $Q(N_1, N_2, N_3, \ldots, N_n, \ldots)$ to represent the number of distinct many-particle states for a given arrangement $(N_1, N_2, N_3, \ldots, N_n, \ldots)$.
- If there are no particles with energy greater than E_m, then for the arrangement $(N_1, N_2, N_3, \ldots, N_n, \ldots)$, we only list the number of particles (N_m) up to and including the highest occupied energy level E_m.
  - For example, $(3, 4)$ denotes that there are three particles in the single-particle states with the lowest energy E_1, four particles in the single-particle states with the first-excited state energy E_2, and zero particles in the single-particle states with higher energy.
- We will use the symbol d_i to represent the degeneracy corresponding to the energy E_i.
  - For example, if $d_i = 5$ then there are five degenerate single-particle states with energy E_i.
- We will ignore the spin degrees of freedom and only consider the spatial part of the wavefunction.

56. Suppose a system with ten single-particle states has 4 particles. The degeneracy of the lowest single-particle stationary states with energy E_1 is $d_1 = 4$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 6$. If the energy of the system is such that 2 particles occupy the lowest single-particle stationary states and 2 particles occupy the first-excited single-particle states, what is the number of distinct four-particle states $Q(2,2)$ corresponding to this particular arrangement $(2,2)$:

(a) if the particles are indistinguishable fermions? Ignore spin.

(b) if the particles are indistinguishable bosons? Ignore spin.

(c) (Hypothetical case) if the identical particles can be treated as distinguishable? Ignore spin.

Consider the following conversation regarding the number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ for a system of identical particles in which the degeneracy of the lowest energy single-particle states with energy E_1 is $d_1 = 4$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 6$.

Student 1: In the given example, since the lowest energy single-particle states with energy E_1 have degeneracy $d_1 = 4$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 6$, there are a total of 10 available single-particle states. We must determine all the permutations of the four particles among the 10 single-particle states.

Student 2: I agree with Student 1 only in the case in which there is no constraint on the total energy of the system. However, in this example, the permutations of the four particles must be consistent with the fixed total energy of the system. Therefore, only two particles with energy E_1 and two particles with energy E_2 are permitted.

Student 3: I agree with Student 2. To determine the number of ways to arrange the two identical particles in the single-particle states with energy E_1, we find the number of ways to arrange the two identical particles when there are four single-particle state available. We can use the following diagram to arrange the two identical particles in the four single-particle states with energy E_1:

E_1	______	______	______	______
	$\psi_{E_1,1}$	$\psi_{E_1,2}$	$\psi_{E_1,3}$	$\psi_{E_1,4}$

Student 2: I agree with Student 3. Similarly to determine the number of ways to arrange the two identical particles in the first-excited single-particle states with energy E_2, we find the number of ways to arrange the two identical particles when there are six single-particle states available. We can use the following diagram to arrange the two identical particles in the six single-particle states with energy E_2:

E_2	______	______	______	______	______	______
	$\psi_{E_2,5}$	$\psi_{E_2,6}$	$\psi_{E_2,7}$	$\psi_{E_2,8}$	$\psi_{E_2,9}$	$\psi_{E_2,10}$

Then combine the number of ways to arrange the particles in the lowest energy single-particle states with the number of ways to arrange the particles in the first-excited single-particle states to find the total number of distinct four-particle states.

Explain why you agree or disagree with each student.

Consider the following three conversations regarding the number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ for a system of indistinguishable fermions in which the degeneracy of the lowest energy single-particle states with energy E_1 is $d_1 = 4$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 6$. Two students consider the number of ways two indistinguishable fermions can be arranged in the lowest energy single-particle states.

Consider the following conversation in which three students consider the number of ways two indistinguishable fermions can be arranged in the lowest energy single-particle states.
Student 1: For the lowest energy single-particle states with energy E_1, which have degeneracy $d_1 = 4$, we must find the number of ways to arrange the two indistinguishable fermions among the four degenerate single-particle states with energy E_1.

E_1 $\quad \psi_{E_1,1} \quad \psi_{E_1,2} \quad \psi_{E_1,3} \quad \psi_{E_1,4}$

Student 2: I agree with Student 1. There are four states in which to arrange the two fermions. Since there can only be one or zero fermions in each degenerate state, there are $\binom{4}{2} = 6$ ways to arrange the two fermions among the lowest energy single-particle states with energy E_1.

Explain why you agree or disagree with each student.

Consider the following conversation in which two students consider the number of ways in which two indistinguishable fermions can be arranged in the first-excited single-particle states.
Student 1: For the first-excited single-particle states with energy E_2 which have degeneracy $d_2 = 6$, we must find the number of ways to arrange the two indistinguishable fermions among the six degenerate single-particle states with energy E_2.

E_2 $\quad \psi_{E_2,5} \quad \psi_{E_2,6} \quad \psi_{E_2,7} \quad \psi_{E_2,8} \quad \psi_{E_2,9} \quad \psi_{E_2,10}$

Student 2: I agree with Student 1. There are six states in which to arrange the two fermions. Since there can only be one or zero fermions in each degenerate state, there are $\binom{6}{2} = 15$ ways to arrange the two fermions among the first-excited single-particle states with energy E_2.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the total number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ for a system of indistinguishable fermions.

Student 1: Since there are 6 ways to arrange two indistinguishable fermions among the four degenerate single-particle states with energy E_1 and 15 ways to arrange two indistinguishable fermions among the six degenerate single-particle states with energy E_2, there are a total of $6+15=21$ distinct four-particle states corresponding to the arrangment of two fermions in the lowest energy states and two fermions in the first-excited states.

Student 2: I disagree with Student 1. The total number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement of two fermions in the lowest energy states and two fermions in the first-excited states is the product of the number of ways to arrange the indistinguishable fermions in the four degenerate states with energy E_1 and the six degenerate states with energy E_2, not the sum. The number of distinct four-particle states corresponding to the arrangement of two fermions in the lowest energy states and two fermions in the first-excited states of the system is $6 \times 15 = 90$.

Do you agree with Student 1 or Student 2? Explain your reasoning.

Consider the following conversation regarding the number of distinct N-particle states $Q(N_1, N_2, N_3, \ldots, N_n, \ldots)$ for a system of indistinguishable fermions in which N_n particles are in the n^{th} single-particle states with energy E_n, which have degeneracy d_n.

Student 1: For each set of degenerate single-particle states, we must find the number of ways to arrange the N_n fermions among the d_n degenerate states. Since each state can contain at most one fermion, the number of ways to choose the N_n occupied states is $\binom{d_n}{N_n}$ in which $N_n \leq d_n$.

Student 2: I agree with Student 1. The total number of distinct N-particle states is the product of the number of ways to arrange the fermions into each single-particle state and is given by

$$\prod_n \frac{d_n!}{N_n!(d_n - N_n)!} = \binom{d_1}{N_1}\binom{d_2}{N_2}\binom{d_3}{N_3}\cdots .$$

Explain why you agree or disagree with the students.

Consider the following two conversations regarding the number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ for a system of indistinguishable bosons in which the degeneracy of the lowest energy single-particle states with energy E_1 is $d_1 = 4$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 6$. Three students consider the number of ways two indistinguishable bosons can be arranged in the lowest energy single-particle states.

Consider the following conversation in which three students consider the number of ways in which two indistinguishable bosons can be arranged among the lowest energy single-particle states.

Student 1: For the lowest energy single-particle states with energy E_1 which have degeneracy $d_1 = 4$, we must find the number of ways to arrange the two indistinguishable bosons among the four degenerate single-particle states with energy E_1.

E_1 _____ $\psi_{E_1,1}$ _____ $\psi_{E_1,2}$ _____ $\psi_{E_1,3}$ _____ $\psi_{E_1,4}$

Student 2: I agree with Student 1. Using the bin and divider method, there are two indistinguishable bosons and three indistinguishable dividers between the four degenerate states. There are five total objects that must be permuted.

Student 3: I agree with both Student 1 and Student 2. When we calculate the number of ways to permute the three indistinguishable dividers with the two bosons, we get $\binom{5}{3} = 10$. There are 10 ways to arrange the two indistinguishable bosons in the lowest energy single-particle states with energy E_1.

Explain why you agree or disagree with the students.

Consider the following conversation in which three students consider the number of ways two indistinguishable bosons can be arranged among the first-excited single-particle states and the total number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ for a system of indistinguishable bosons.

Student 1: For the first-excited single-particle states with energy E_2 which have degeneracy $d_2 = 6$, we must find the number of ways to arrange the two indistinguishable bosons among the six degenerate single-particle states with energy E_2.

E_2 _____ $\psi_{E_2,5}$ _____ $\psi_{E_2,6}$ _____ $\psi_{E_2,7}$ _____ $\psi_{E_2,8}$ _____ $\psi_{E_2,9}$ _____ $\psi_{E_2,10}$

Student 2: I agree with Student 1. Using the bin and divider method, there are two indistinguishable bosons and five indistinguishable dividers between the six degenerate states. There are seven total objects to be permuted, two indistinguishable bosons and five indistinguishable dividers. When we calculate the number of ways to permute the five indistinguishable dividers with the two bosons, we get $\binom{7}{2} = 21$.

Student 3: I agree with both Student 1 and Student 2. There are 10 ways to arrange the two indistinguishable bosons among the lowest stationary states with energy E_1 and 21 ways to arrange the two indistinguishable bosons among the first-excited single-particle states with energy E_2. The total number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ is $10 \cdot 21 = 210$.

Explain why you agree or disagree with the students.

Consider the following conversation regarding the number of distinct N-particle states $Q(N_1, N_2, N_3, \ldots, N_n, \ldots)$ for a system of indistinguishable bosons in which N_n particles are in the n^{th} single-particle states with energy E_n, which has degeneracy d_n.

Student 1: For each set of degenerate single-particle states, we must find the number of ways to arrange the N_n bosons among the d_n degenerate states. Using the bin and divider method, there are N_n indistinguishable bosons and $d_n - 1$ indistinguishable dividers between the d_n degenerate states. There are $N_n + d_n - 1$ total objects that must be permuted. When we calculate the number of ways to permute the $d_n - 1$ indistinguishable dividers with the N_n bosons, we get $\binom{N_n+d_n-1}{d_n-1} = \binom{N_n+d_n-1}{N_n}$.

Student 2: I agree with Student 1. The total number of distinct N-particle states is the product of the ways to arrange the bosons into each group of degenerate single-particle states and is given by

$$\prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} = \binom{N_1 + d_1 - 1}{d_1 - 1}\binom{N_2 + d_2 - 1}{d_2 - 1}\binom{N_3 + d_3 - 1}{d_3 - 1}\cdots.$$

Explain why you agree or disagree with the students.

Hypothetical Case: Treating the identical particles as distinguishable.

Consider the following two conversations regarding the number of distinct four-particle states $Q(2, 2)$ corresponding to the arrangement $(2, 2)$ for a system of identical particles which can be treated as distinguishable, in which the degeneracy of the lowest energy single-particle states with energy E_1 is $d_1 = 4$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 6$. Three students consider the number of ways two identical particles can be arranged in the lowest energy single-particle states if they can be treated as distinguishable.

In the following conversation three students consider the number of ways two distinguishable particles can be arranged among the lowest energy single-particle states.

Student 1: For the lowest energy single-particle states with energy E_1 which has degeneracy $d_1 = 4$, we must find the number of ways to arrange the two distinguishable particles among the four degenerate single-particle states with energy E_1.

E_1 ______ $\psi_{E_1,1}$ ______ $\psi_{E_1,2}$ ______ $\psi_{E_1,3}$ ______ $\psi_{E_1,4}$

Student 2: I agree with Student 1. Since the particles can be treated as distinguishable, we need to take into account which particles we are choosing, i.e., there are $\binom{4}{2}= 6$ different two particle combinations to arrange in the lowest energy single-particle states with energy E_1. Within the 4-fold degenerate lowest energy single-particle states, there are four degenerate single-particle states available to the first particle and four degenerate single-particle states for the second particle. There are 4^2 ways to arrange the two particles.

Student 3: I agree with both Student 1 and Student 2. There is a total of $6 \cdot 16 = 96$ ways to arrange two of the four identical particles which can be treated as distinguishable in the lowest energy single-particle states.

Explain why you agree or disagree with the students.

Consider the following conversation in which three students consider the number of ways two distinguishable particles can be arranged among the first-excited single-particle states and the total number of distinct four-particle states $Q(2,2)$ corresponding to the arrangement $(2,2)$ for a system of distinguishable particles.
Student 1: For the set of degenerate first-excited single-particle states with energy E_2 which has degeneracy $d_2 = 6$, we must find the number of ways to arrange the two distinguishable particles among the six degenerate single-particle states with energy E_2.

E_2

$\psi_{E_2,5}$ $\psi_{E_2,6}$ $\psi_{E_2,7}$ $\psi_{E_2,8}$ $\psi_{E_2,9}$ $\psi_{E_2,10}$

Student 2: I agree with Student 1. If the **particles can be treated as distinguishable**, we need to take into account which particles we are choosing. Since we chose two particles for the lowest energy single-particle states, there are two identical particles remaining for the first-excited single-particle states. There is only $\binom{4-2}{2} = \binom{2}{2} = 1$ two particle combination to arrange among the first-excited single-particle states with energy E_2. Within the 6-fold degenerate first-excited single-particle states, there are six degenerate single-particle states available to the first particle and six degenerate single-particle states for the second particle. There are $6^2 = 36$ ways to arrange the two particles.
Student 3: I agree with both Student 1 and Student 2. There are 96 ways to arrange two particles among the lowest energy single particle stationary states first and 36 ways to arrange the remaining two particles among the first-excited single-particle states. The number of distinct four-particle states corresponding to the arrangement $(2,2)$ for a system of identical particles which can be treated as distinguishable is $96 \cdot 36 = 3456$.

Explain why you agree or disagree with the students.

Consider the following conversation regarding the number of distinct N-particle states $Q(N_1, N_2, N_3, \ldots, N_n, \ldots)$ for a system of N identical particles which can be treated as distinguishable in which N_n particles are in the d_n-fold degenerate single-particle states with energy E_n.

Student 1: To determine the number of distinct N-particle states for a system of N **identical particles which can be treated as distinguishable** in which N_n particles are in the d_n-fold degenerate single-particle states with energy E_n, we can first choose which of the N particles are in the set of degenerate states with energy E_n and then multiply by the number of ways to arrange the particles among the single-particle states.

Student 2: I agree with Student 1. If there are N_1 particles in the d_1-fold degenerate lowest stationary state, then there are $\binom{N}{N_1}$ ways to choose the N_1 particles in the lowest stationary state and there are $d_1^{N_1}$ ways to arrange the N_1 particles among the d_1 degenerate lowest single-particle states.

Student 3: I agree with Student 2. If there are N_2 particles in the d_2-fold degenerate first-excited single-particle states, then there are $N - N_1$ particles from which to choose the N_2 particles in the first-excited single-particle states. Then, there are d_2 states available to the N_2 particles so there are $d_2^{N_2}$ ways to arrange the particles in the first-excited single-particle states.

Student 1: I agree with both Student 2 and Student 3. We can continue this way and the total number of distinct N-particle states for a system of N identical particles which can be treated as distinguishable is

$$\begin{aligned}
&\left[\binom{N}{N_1} d_1^{N_1}\right] \cdot \left[\binom{N - N_1}{N_2} d_2^{N_2}\right] \cdot \left[\binom{N - N_1 - N_2}{N_3} d_3^{N_3}\right] \cdots \\
&= \left[\frac{N!}{N_1!(N - N_1)!} d_1^{N_1}\right] \cdot \left[\frac{(N - N_1)!}{N_2!(N - N_1 - N_2)!} d_2^{N_2}\right] \cdot \left[\frac{(N - N_1 - N_2)!}{N_3!(N - N_1 - N_2 - N_3)!} d_3^{N_3}\right] \cdots \\
&= N! \frac{d_1^{N_1} d_2^{N_2} d_3^{N_3} \cdots}{N_1! N_2! N_3! \cdots} \\
&= N! \prod_n \frac{d_n^{N_n}}{N_n!}
\end{aligned}$$

Explain why you agree or disagree with each student.

Summary for Determining the Number of Distinct Many-Particle States when the Total Energy of the Many-Particle System is Fixed and the Single-Particle States have Degeneracy

To summarize what you have learned about determining the number of distinct many-particle states for a many-particle system with fixed energy and in which the single-particle states have degeneracy, answer the following question.

57. Suppose that a system with six single-particle states has 6 particles. The degeneracy of the lowest single-particle states with energy E_1 is $d_1 = 3$ and the degeneracy of the first-excited single-particle states with energy E_2 is $d_2 = 3$. If the system has the arrangement $(2, 4)$ such that 2 particles are in the lowest single-particle states and 4 particles are in the first-excited single-particle states, what is the number of distinct six-particle states $Q(2, 4)$ corresponding to this particular arrangement $(2, 4)$:

    (a) if the particles are indistinguishable fermions? Ignore spin.

    (b) if the particles are indistinguishable bosons? Ignore spin.

    (c) (Hypothetical case) if the identical particles can be treated as distinguishable? Ignore spin.

Compare the number of distinct four-particle states $Q(2, 4)$ for the cases in which the 6 particles are indistinguishable fermions, indistinguishable bosons, and the hypothetical case in which particles can be treated as distinguishable particles.

****CHECKPOINT: Check your answers to questions 56-57. ****

56a. $\binom{4}{2} \cdot \binom{6}{2} = 6 \times 15 = 90$

or equivalently $\prod_n \frac{d_n!}{N_n!(d_n - N_n)!} = \left(\frac{4!}{2!(4-2)!}\right)\left(\frac{6!}{2!(6-2)!}\right) = 90$

56b. $\binom{2+4-1}{2} \cdot \binom{2+6-1}{2} = \binom{5}{2} \cdot \binom{7}{2} = 10 \times 21 = 210$

or equivalently $\prod_n \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} = \left(\frac{(2+4-1)!}{2!(4-1)!}\right)\left(\frac{(2+6-1)!}{2!(6-1)!}\right) = 210$

56c. $\left[\binom{4}{2} \cdot 4^2\right]\left[\binom{4-2}{2} \cdot 6^2\right] = 96 \times 36 = 3456$

or equivalently $N! \prod_n \frac{{d_n}^{N_n}}{N_n!} = 4!\left(\frac{4^2}{2!}\right)\left(\frac{6^2}{2!}\right) = 3456$

57a. 0. There cannot be four fermions in the second single-particle state with energy E_2 since it has degeneracy $d_2 = 3$. There must at least as many available states as the number of fermions.

57b. $\binom{2+3-1}{2} \cdot \binom{4+3-1}{4} = \binom{4}{2} \cdot \binom{6}{4} = 6 \times 15 = 90$

or equivalently $\prod_n \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} = \left(\frac{(2+3-1)!}{2!(3-1)!}\right)\left(\frac{(4+3-1)!}{4!(3-1)!}\right) = 90$

57c. $\left[\binom{6}{2} \cdot 3^2\right]\left[\binom{6-2}{4} \cdot 3^4\right] = [15 \cdot 9][1 \cdot 81] = 10,935$

or equivalently $N! \prod_n \frac{{d_n}^{N_n}}{N_n!} = 6!\left(\frac{3^2}{2!}\right)\left(\frac{3^4}{4!}\right) = 10,935$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

7 Stationary State Wavefunction for a System of N Non-Interacting Particles (Including Spin)

- When considering the spin part of the wavefunction for a single-particle, we will use the notation $|s_i,\ m_{s_i}\rangle$ (in which s_i and m_{s_i} are the quantum numbers corresponding to the total spin and z-component of the spin for the i^{th} particle, respectively).
  - The states $|s_1,\ m_{s_1}\rangle$ are eigenstates of $\hat{S}_1^2$ and $\hat{S}_{1z}$ and the states $|s_2,\ m_{s_2}\rangle$ are eigenstates of $\hat{S}_2^2$ and $\hat{S}_{2z}$.
- When considering the spin part of the wavefunction for the two particles in the **uncoupled representation** in the product space, we will use the notation $|s_1,\ m_{s_1}\rangle_1|s_2,\ m_{s_2}\rangle_2$ for the basis states.
- Unless otherwise specified, we will consider only systems of spin-1/2 particles confined in one spatial dimension.
- Even though the spatial and spin parts of the wavefunction can be entangled in many situations, we will only consider separable many-particle wavefunctions in one-dimension that can be written as the product of the spatial part of the wavefunction $\psi(x_1, x_2, x_3, \ldots)$ and the spin part of the wavefunction $\chi(m_{s_1}, m_{s_2}, m_{s_3}, \ldots)$

$$\Psi(x_1, x_2, x_3, \ldots, m_{s_1}, m_{s_2}, m_{s_3}, \ldots) = \psi(x_1, x_2, x_3, \ldots)\chi(m_{s_1}, m_{s_2}, m_{s_3}, \ldots)$$

Recall: The eigenstates of the z-component of spin for a spin-1/2 system $|s_i\ m_{s_i}\rangle_i$ can be $\left\{\left|\frac{1}{2},\ \frac{1}{2}\right\rangle_i, \left|\frac{1}{2},\ -\frac{1}{2}\right\rangle_i\right\}$ (since for $s_i = \frac{1}{2}$, $m_{s_i} = \frac{1}{2}$ or $-\frac{1}{2}$). For a system of two spin-1/2 particles, e.g. electrons, we will use the following notation for the spin state of each particle since it can have spin "up" or spin "down":

$$\text{Spin "Up"} \quad |\uparrow\rangle_i = \left|\frac{1}{2},\ \frac{1}{2}\right\rangle_i \qquad \text{Spin "Down"} \quad |\downarrow\rangle_i = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_i$$

- When considering the spin part of the wavefunction for the two spin-1/2 particles ($s_1 = 1/2 \otimes s_2 = 1/2$) in the **uncoupled representation** in the product space, we will use the notation $|\uparrow\rangle_1|\uparrow\rangle_2$, $|\uparrow\rangle_1|\downarrow\rangle_2$, $|\downarrow\rangle_1|\uparrow\rangle_2$, and $|\downarrow\rangle_1|\downarrow\rangle_2$ for the basis states.
- We will also use the notation in the **coupled representation** $|s,\ m_s\rangle$ in which the quantum numbers s and m_s correspond to the total spin angular momentum and the z component of the total spin angular momentum including both spins, respectively (we will use the notation that a state in the coupled representation will not have a subscript whereas states in the uncoupled representation will have a subscript indicating the particle associated with each spin state).
  - The states $|s,\ m_s\rangle$ in the coupled representation are eigenstates of $\hat{S}^2$ and $\hat{S}_z$ where $\vec{S} = \vec{S}_1 + \vec{S}_2$.
- For a system of two spin-1/2 particles ($s_1 = 1/2 \otimes s_2 = 1/2$), the total spin quantum number $s = s_1 + s_2 = 1/2 + 1/2 = 1$ or $s = |s_1 - s_2| = |1/2 - 1/2| = 0$.
  - If the total spin quantum number is $s = 1$ then the corresponding $m_s = -1, 0, 1$ and the states in the coupled representation are given by $|s, m_s\rangle = \{|1,\ 1\rangle, |1,\ 0\rangle, |1,\ -1\rangle\}$. If $s = 0$, then the corresponding $m_s = 0$ and the state in the coupled representation is given by $|s, m_s\rangle = |0,\ 0\rangle$.
- We will use the following abbreviated notation for a complete set of normalized states for a system of two spin-1/2 particles in the coupled representation $|s,\ m_s\rangle$ written in terms of the uncoupled representation.

$$\boxed{\begin{array}{lcccc} |1,\ 1\rangle & = & |\uparrow\uparrow\rangle & = & |\uparrow\rangle_1|\uparrow\rangle_2 \\ |1,\ -1\rangle & = & |\downarrow\downarrow\rangle & = & |\downarrow\rangle_1|\downarrow\rangle_2 \\ |1,\ 0\rangle & = & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & = & \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2) \\ |0,\ 0\rangle & = & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) & = & \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2) \end{array}}$$

- If you are not familiar with the formalism of addition of angular momentum (including how to write a complete set of basis states in the coupled and uncoupled representations or how to write various operators in the coupled and uncoupled representations), please work through the pretest, warm-up, tutorial and posttest for the Addition of Angular Momentum tutorial (since it would help you in writing the spin part of the many-particle state in a particular representation).

58. For the spin part of the wavefunction (spin state) of a two-particle system ($s_1 = 1/2 \otimes s_2 = 1/2$) given below in the uncoupled representation, identify whether the spin state is symmetric, antisymmetric, or neither symmetric nor antisymmetric with respect to exchange of the two particles. Labels 1 and 2 denote particles 1 and 2, respectively. Explain your reasoning.

(a) $|\uparrow\rangle_1|\uparrow\rangle_2$

(b) $|\downarrow\rangle_1|\downarrow\rangle_2$

(c) $C_1|\uparrow\rangle_1|\uparrow\rangle_2 + C_2|\downarrow\rangle_1|\downarrow\rangle_2$ (with $C_1 \neq C_2$ and $|C_1|^2 + |C_2|^2 = 1$)

(d) $|\uparrow\rangle_1|\uparrow\rangle_2 - |\downarrow\rangle_1|\downarrow\rangle_2$

(e) $|\uparrow\rangle_1|\downarrow\rangle_2$

(f) $|\downarrow\rangle_1|\uparrow\rangle_2$

(g) $C_1|\uparrow\rangle_1|\downarrow\rangle_2 + C_2|\downarrow\rangle_1|\uparrow\rangle_2$ (with $C_1 \neq \pm C_2$ and $|C_1|^2 + |C_2|^2 = 1$)

(h) $C_1|\uparrow\rangle_1|\uparrow\rangle_2 + C_2|\downarrow\rangle_1|\downarrow\rangle_2 + \frac{C_3}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2)$ (with $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$)

59. Based on your answer to 58, in the uncoupled representation ($s_1 = 1/2 \otimes s_2 = 1/2$), are the spin states $|\uparrow\rangle_1|\uparrow\rangle_2$, $|\downarrow\rangle_1|\downarrow\rangle_2$, $|\uparrow\rangle_1|\downarrow\rangle_2$, $|\downarrow\rangle_1|\uparrow\rangle_2$, $|\uparrow\rangle_1|\uparrow\rangle_2 + |\downarrow\rangle_1|\downarrow\rangle_2$ and $|\uparrow\rangle_1|\uparrow\rangle_2 - |\downarrow\rangle_1|\downarrow\rangle_2$ an appropriate spin part of the wavefunction for a system of two indistinguishable spin-1/2 particles for writing a completely symmetric/antisymmetric wavefunction? Explain your reasoning.

60. For the spin part of the wavefunction (spin state) for ($s_1 = 1/2 \otimes s_2 = 1/2$) of a two-particle system given below in the coupled representation expressed in terms of the uncoupled representation, identify whether the spin state is symmetric, antisymmetric, or neither symmetric nor antisymmetric with respect to exchange of the two particles. Explain your reasoning.

(a) $|1,\ 1\rangle = |\uparrow\uparrow\rangle$

(b) $|1,\ -1\rangle = |\downarrow\downarrow\rangle$

(c) $|1,\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

(d) $|0,\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

(e) $C_1|1,\ 0\rangle + C_2|0,\ 0\rangle = \frac{C_1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + \frac{C_2}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ (with $|C_1|^2 + |C_2|^2 = 1$)

(f) $C_1|1,\ 1\rangle + C_2|1,\ -1\rangle + C_3|1,\ 0\rangle = C_1|\uparrow\uparrow\rangle + C_2|\downarrow\downarrow\rangle + \frac{C_3}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ (with $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$)

61. Based on your answer to question 60, in the coupled representation, are the spin states $|1,\ 1\rangle$, $|1,\ -1\rangle$, $|1,\ 0\rangle$, $|0,\ 0\rangle$, and $\frac{1}{\sqrt{3}}[|1,\ 1\rangle + |1,\ -1\rangle + |0,\ 0\rangle]$ an appropriate spin part of the wavefunction for a system of two indistinguishable spin-1/2 particles for writing a completely symmetric/antisymmetric wavefunction? Explain your reasoning.

Consider the following conversation regarding whether a spin state in the coupled representation is symmetric or antisymmetric for a system of two spin-1/2 particles ($s_1 = 1/2 \otimes s_2 = 1/2$).

Student 1: The spin state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2\rangle - |\downarrow\rangle_1|\uparrow\rangle_2)$ is symmetric since exchanging the particles results in the same spin state.

Student 2: I disagree with Student 1. The spin state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2\rangle - |\downarrow\rangle_1|\uparrow\rangle_2)$ is antisymmetric. If we exchange the particles, we get $\frac{1}{\sqrt{2}}(|\uparrow\rangle_2|\downarrow\rangle_1) - |\downarrow\rangle_2|\uparrow\rangle_1) = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) = -\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Student 3: I agree with Student 2. The antisymmetric spin state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is referred to as the "singlet" state since it corresponds to the total spin quantum number $s = 0$ for a system of two spin-1/2 particles for which the only possibility for m_s is $m_s = 0$.

Explain why you agree or disagree with each student.

Consider the following conversation regarding whether a spin state in the coupled representation for a system of two spin-1/2 particles ($s_1 = 1/2 \otimes s_2 = 1/2$) is symmetric or antisymmetric.

Student 1: The spin state $|\uparrow\uparrow\rangle = |\uparrow\rangle_1|\uparrow\rangle_2$ is symmetric since exchanging the two particles results in the same spin state $|\uparrow\uparrow\rangle = |\uparrow\rangle_2|\uparrow\rangle_1$. Similarly, the spin state $|\downarrow\downarrow\rangle = |\downarrow\rangle_1|\downarrow\rangle_2$ is symmetric.

Student 2: I agree with Student 1. The spin state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2\rangle + |\downarrow\rangle_1|\uparrow\rangle_2)$ is also symmetric since exchanging the two particles results in the same spin state.

Student 3: The spin states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ are all symmetric and referred to as the "triplet" states since they correspond to the total spin quantum number $s = 1$ for a system of two spin-1/2 particles with $m_s = +1, -1, 0$, respectively.

Explain why you agree or disagree with the students.

Consider the following conversation regarding choosing states for a system of two spin-1/2 particles with regard to symmetrization requirements.

Student 1: In the uncoupled representation, the two-particle spin states $|\uparrow\rangle_1|\uparrow\rangle_2$, $|\downarrow\rangle_1|\downarrow\rangle_2$, $|\uparrow\rangle_1|\downarrow\rangle_2$, and $|\downarrow\rangle_1|\uparrow\rangle_2$ are all appropriate choices for spin part of the wavefunction to satisfy the symmetrization requirement.

Student 2: I disagree with Student 1. In order to satisfy the symmetrization requirement of the wavefunction, we must choose spin states which are either symmetric or antisymmetric. In the uncoupled representation, the two-particle spin states $|\uparrow\rangle_1|\downarrow\rangle_2$ and $|\downarrow\rangle_1|\uparrow\rangle_2$ are neither symmetric nor antisymmetric. For example, exchanging particles 1 and 2 transforms the state $|\uparrow\rangle_1|\downarrow\rangle_2$ to $|\uparrow\rangle_2|\downarrow\rangle_1 = |\downarrow\rangle_1|\uparrow\rangle_2$ but $|\uparrow\rangle_1|\downarrow\rangle_2 \neq \pm|\downarrow\rangle_1|\uparrow\rangle_2$ so $|\uparrow\rangle_1|\downarrow\rangle_2$ is neither symmetric nor antisymmetric. The same is true for the spin state $|\downarrow\rangle_1\uparrow\rangle_2$.

Student 3: I agree with Student 2. The two-particle spin states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ in the coupled representation expressed in terms of states in the uncoupled representation, are symmetric. The two-particle spin state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is antisymmetric. Therefore, the two-particle spin states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, and $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ are all appropriate choices for spin part of the two-particle wavefunction with suitable spatial wavefunction to satisfy the symmetrization requirement.

Explain why you agree or disagree with each student.

62. Write four possible two-particle wavefunctions including spin for a system of two non-interacting indistinguishable fermions in single-particle states labeled by n_1 and n_2.

Consider the following conversation regarding constructing a completely antisymmetric wavefunction for a system of indistinguishable non-interacting fermions.

Student 1: For a system of two fermions, the two-particle wavefunction, which is made up of the product of the spatial part and spin part of the wavefunction, must be antisymmetric.

Student 2: I disagree with Student 1. We must only ensure that the spatial part of the two-particle wavefunction is antisymmetric. The spatial part of the two-particle stationary state wavefunction must be $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$. The spin part of the two-particle wavefunction can be either the antisymmetric singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ or one of the three symmetric triplet states $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\}$.

Student 3: I agree with Student 2 that the spatial part of the two-particle wavefunction must be antisymmetric. However, we must also choose the antisymmetric singlet state as the spin part of the two-particle wavefunction.

Student 4: I disagree with both Student 2 and Student 3. The overall two-particle wavefunction must be antisymmetric. If the spatial part of the two-particle wavefunction is symmetric $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$, the spin part of the two-particle wavefunction must be the antisymmetric singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Student 1: I agree with Student 4. Additionally, the spatial part of the two-particle wavefunction could be antisymmetric $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$, which would imply that the spin part of the two-particle wavefunction can be one of the symmetric triplet states $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\}$. In either case, the product of one symmetric and one antisymmetric wavefunction produces an overall antisymmetric two-particle wavefunction.

Student 4: I agree with Student 1. However, remember that a linear combination of the triplet states such as $C_1|\uparrow\uparrow\rangle + C_2|\downarrow\downarrow\rangle + \frac{C_3}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ is a completely symmetric spin state. This state is normalized if we choose $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$.

Explain why you agree or disagree with each student.

63. Fill in all the possibilities in the table below based on what you learned about the symmetric/antisymmetric characteristic (with respect to exchange of two particles) of the many-particle wavefunction for a system of identical particles.

Type of Particle	Spatial Part of the Many-Particle Wavefunction (Symmetric/Antisymmetric)	Spin part of the Many-Particle Wavefunction (Symmetric/Antisymmetric)	Complete Many-Particle Wavefunction (Symmetric/Antisymmetric)
Indistinguishable Fermions			
Indistinguishable Bosons			

****CHECKPOINT: Check your answers to questions 58-63. ****

58a. Symmetric
58b. Symmetric
58c. Symmetric
58d. Symmetric
58e. Neither symmetric nor antisymmetric
58f. Neither symmetric nor antisymmetric
58g. Neither symmetric nor antisymmetric
58h. Symmetric
59. The spin states $|\uparrow\rangle_1|\downarrow\rangle_2$ and $|\downarrow\rangle_1|\uparrow\rangle_2$ are neither symmetric nor antisymmetric. It is not possible to combine either of these two spin states individually with the spatial part of the wavefunction to produce a wavefunction that is either completely symmetric or completely antisymmetric.
The spin states $|\uparrow\rangle_1|\uparrow\rangle_2$, $|\downarrow\rangle_1|\downarrow\rangle_2$, and $|\uparrow\rangle_1|\uparrow\rangle_2 + |\downarrow\rangle_1|\downarrow\rangle_2$ are symmetric and the spin state $|\uparrow\rangle_1|\uparrow\rangle_2 - |\downarrow\rangle_1|\downarrow\rangle_2$ is antisymmetric and could be combined with the spatial part of the wavefunction to produce a wavefunction that is either completely symmetric or completely antisymmetric.
60a. Symmetric
60b. Symmetric
60c. Symmetric
60d. Antisymmetric
60e. Neither symmetric nor antisymmetric
60f. Symmetric
61. The spin states $C_1|1,\ 0\rangle + C_2|0,\ 0\rangle$ is neither symmetric nor antisymmetric. It is not possible to combine this spin states individually with the spatial part of the wavefunction to produce a wavefunction that is either completely symmetric or completely antisymmetric.
The spin states $|1,\ 1\rangle$, $|1,\ -1\rangle$, $|1,\ 0\rangle$, and $C_1|1,\ 1\rangle + C_2|1,\ -1\rangle + C_3|1,\ 0\rangle$ in the coupled representation are symmetric. The spin state $|0,\ 0\rangle$ is antisymmetric. Therefore it is possible to combine these spin states with the spatial part of the wavefunction to produce a wavefunction that is either completely symmetric or completely antisymmetric.
62. The following are examples of a two-particle wavefunction including spin for a system of two non-interacting indistinguishable fermions in single-particle states labeled by n_1 and n_2

$$\begin{aligned}
\Psi(x_1, x_2, m_{s_1}, m_{s_2}) &= [\tfrac{1}{\sqrt{2}}\{\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\}][|\uparrow\uparrow\rangle] \\
\Psi(x_1, x_2, m_{s_1}, m_{s_2}) &= [\tfrac{1}{\sqrt{2}}\{\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\}][|\downarrow\downarrow\rangle] \\
\Psi(x_1, x_2, m_{s_1}, m_{s_2}) &= [\tfrac{1}{\sqrt{2}}\{\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\}][\tfrac{1}{\sqrt{2}}\{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\}] \\
\Psi(x_1, x_2, m_{s_1}, m_{s_2}) &= [\tfrac{1}{\sqrt{2}}\{\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\}][\tfrac{1}{\sqrt{2}}\{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\}] \\
\\
\Psi(x_1, x_2, m_{s_1}, m_{s_2}) &= [\tfrac{1}{\sqrt{2}}\{\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\}][\tfrac{1}{\sqrt{2}}\{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}]
\end{aligned}$$

63.

Type of Particle	Spatial Part of the Wavefunction (Symmetric/ Antisymmetric)	Spin part of the Wavefunction (Symmetric/ Antisymmetric)	Complete Wavefunction (Symmetric/ Antisymmetric)
Indistinguishable Fermions	Symmetric	Antisymmetric	Antisymmetric
	Antisymmetric	Symmetric	
Indistinguishable Bosons	Symmetric	Symmetric	Symmetric
	Antisymmetric	Antisymmetric	

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

64. Consider a system with three identical non-interacting spin-1/2 particles. If two of the particles are in the spin up state and one of the particles is in the spin down state, construct a completely symmetric spin state for the three particle system. If no such spin state exists, state the reason why. (Hint: Start with the basis state $|\uparrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3$.)

65. Consider a system with three identical non-interacting spin-1/2 particles. If two of the particles are in the spin up state and one of the particles is in the spin down state, construct a completely antisymmetric spin state for the three particle system. If no such spin state exists, state the reason why. (Hint: Start with the basis state $|\uparrow\rangle_1 |\uparrow\rangle_2 |\downarrow\rangle_3$.)

64. $\frac{1}{\sqrt{3}}[|\uparrow\rangle_1|\uparrow\rangle_2|\downarrow\rangle_3 + |\uparrow\rangle_1|\downarrow\rangle_2|\uparrow\rangle_3 + |\downarrow\rangle_1|\uparrow\rangle_2|\uparrow\rangle_3]$
65. It is not possible to construct a completely antisymmetric spin state for a system with two particles in the same spin state.

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

8 Examples of Finding the Many-Particle Stationary State Wavefunctions and Energies (including spin)

In this section and the next, we shall focus on determining the many-particle stationary state wavefunction for a system of non-interacting particles placed in a one-dimensional harmonic oscillator potential well. Previously, we considered the many-particle stationary state wavefunction for a system of non-interacting particles placed in an infinite square well potential. Take a moment to think about the form of the many-particle stationary state wavefunction for a system of identical fermions or bosons in these two systems and whether the different potential energy terms affect the form of the many-particle stationary state wavefunction.

8.1 One-Dimensional Harmonic Oscillator - Two Spin-1/2 Fermions

Two identical non-interacting spin-1/2 fermions are placed in a one-dimensional harmonic oscillator potential energy well with Hamiltonian $\hat{H}_i = \frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_i{}^2$. The single-particle energies are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots$$

For the following questions, you can denote the spatial state of the i^{th} particle in the $n_i{}^{th}$ single-particle state of the oscillator by $\psi_{n_i}(x_i)$.

66. Find the two-particle <u>ground state</u> and <u>first-excited state</u> energies of the two-particle system if the particles are

    (a) Indistinguishable fermions with spin-1/2 in a total spin $s = 0$ state.

    (b) Indistinguishable fermions with spin-1/2 in a total spin $s = 1$ state.

67. Construct the spatial part of the two-particle <u>ground state</u> and <u>first-excited state</u> for two non-interacting particles in the one-dimensional harmonic oscillator potential energy well if the particles are

    (a) Indistinguishable fermions with spin-1/2 in a total spin $s = 0$ state.

    (b) Indistinguishable fermions with spin-1/2 in a total spin $s = 1$ state.

Consider the following conversation regarding the two-particle ground state and ground state energy for two indistinguishable fermions with spin-1/2 in a total spin $s = 0$ state placed in a one-dimensional harmonic oscillator potential energy well.

Student 1: For the two-particle ground state for a system of two indistinguishable fermions with spin-1/2 in a total spin $S = 0$ state, both fermions are in the single-particle spatial state ψ_0, so $n_1 = n_2 = 0$. The many-particle ground state energy is $E_{00} = E_0 + E_0 = \hbar\omega$.

Student 2: I disagree with Student 1. The two fermions cannot both be in the same single-particle spatial state ψ_0. For the two-particle ground state, one fermion is in the lowest single-particle spatial state ψ_0 and the other fermion is in the first-excited single-particle spatial state ψ_1, so $n_1 = 0$ and $n_2 = 1$ or $n_1 = 1$ and $n_2 = 0$. The two-particle ground state energy is $E_{10} = E_1 + E_0 = 2\hbar\omega$.

Student 3: I agree with Student 1 and disagree with Student 2. For a system of indistinguishable fermions, the overall two-particle state must be antisymmetric. Since the fermions are in the total spin $s = 0$ antisymmetric singlet state $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, the spatial part of the many-particle state must be symmetric. Two fermions in the same single-particle spatial state ψ_0 correspond to the symmetric spatial state $\psi_0(x_1)\psi_0(x_2)$

Student 1: I agree with Student 3. The overall two-particle ground state including both spatial and spin parts is $\Psi_{00} = [\psi_0(x_1)\psi_0(x_2)][\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)]$. In the total spin $s = 0$ state, the two fermions can be in the same single-particle spatial state ψ_0 since the fermions are in different spin states with the two-particle spin-state $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ being antisymmetric.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the two-particle first-excited state and first-excited state energy for two indistinguishable fermions with spin-1/2 in a total spin $s = 0$ state placed in a one-dimensional harmonic oscillator potential energy well.

Student 1: For a system of indistinguishable fermions, the overall two-particle state must be antisymmetric. Since the fermions are in the total spin $s = 0$ antisymmetric singlet state $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, the spatial part of the two-particle state must be symmetric.

Student 2: In the two-particle first-excited spatial state for a system of two indistinguishable fermions with spin-1/2 in a total spin $s = 0$ state, one fermion is in the lowest single-particle spatial state ψ_0 and the other fermion is in the first-excited single-particle spatial state ψ_1, so $n_1 = 1$ and $n_2 = 0$ or $n_1 = 0$ and $n_2 = 1$. The two-particle first-excited state energy is $E_{10} = E_1 + E_0 = 2\hbar\omega$.

Student 3: I agree with Student 1 and Student 2. The spatial part of the two-particle first-excited state is symmetric and given by $\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2))$. The overall two-particle first-excited state including both spatial and spin parts is $\Psi_{01} = [\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2))][\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)]$.

Explain why you agree or disagree with the students.

Consider the following conversation regarding the two-particle ground state and ground state energy for two indistinguishable fermions with spin-1/2 in a total spin $s = 1$ state placed in a one-dimensional harmonic oscillator potential energy well.

Student 1: For the two-particle ground state for a system of two indistinguishable fermions with spin-1/2 in a total spin $S = 1$ state, both fermions are in the single-particle spatial state ψ_0, so $n_1 = n_2 = 0$. The two-particle ground state energy is $E_{00} = \hbar\omega$.

Student 2: I disagree. For a system of indistinguishable fermions, the overall two-particle state including both spatial and spin parts must be antisymmetric. Since the fermions are in a total spin $s = 1$ symmetric triplet state, the spatial part of the two-particle state must be antisymmetric. The two fermions cannot be in the same single-particle spatial state ψ_0 because that is a symmetric state.

Student 3: I agree with Student 2. The two-particle ground state must include the antisymmetric spatial state in which one fermion is in the single-particle state ψ_0 and the other fermion is in the single-particle spatial state ψ_1, so $n_1 = 1$ and $n_2 = 0$ or $n_1 = 0$ and $n_2 = 1$.

Student 2: Right! The antisymmetric spatial part of the two-particle ground state is $\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2))$. One possible two-particle ground state including both spatial and spin parts is $\Psi_{00} = [\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2))][|\uparrow\uparrow\rangle]$. The two-particle ground state energy is $E_{10} = 2\hbar\omega$.

Student 3: I agree with Student 2. Additionally, if the spatial part of two-particle ground state is $\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2))$, then the spin part of the wavefunction could be $|\downarrow\downarrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, or $C_1|\uparrow\rangle + C_2|\downarrow\rangle + \frac{C_3}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ in which $|C_1|^2 + |C_2|^2 + |C_3|^2 = 1$.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the two-particle first-excited state and first-excited state energy for two indistinguishable fermions with spin-1/2 in a total spin $s = 1$ state placed in a one-dimensional harmonic oscillator potential energy well.

Student 1: The two-particle first-excited state energy for two spin-1/2 fermions in a total spin $s = 1$ state is $E_{11} = 3\hbar\omega$, in which both fermions are in the same single-particle spatial state ψ_1.

Student 2: I disagree. In the total spin $s = 1$ state, both fermions are in the same spin state and therefore cannot be in the same single-particle spatial state ψ_1.

Student 3: I disagree with Student 1's reasoning. Since the fermions are in a total spin $s = 1$ symmetric triplet state, the spatial part of the two-particle state must be antisymmetric so that the overall two-particle state is antisymmetric. The two fermions cannot be in the same spatial state ψ_1 because this would mean that both the spatial part and spin part of the wavefunction are symmetric which is not allowed. I disagree with Student 2's reasoning, as it does not hold for the triplet state $|\chi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 + |\uparrow\rangle_2|\downarrow\rangle_1)$.

Student 4: In the two-particle first-excited state for a system of two indistinguishable fermions with spin-1/2 in a total spin $s = 1$ state, one fermion is in the single-particle spatial state ψ_0 and the other fermion is in the single-particle spatial state ψ_2, so $n_1 = 2$ and $n_2 = 0$ or $n_1 = 0$ and $n_2 = 2$. The spatial part of the two-particle first-excited state is antisymmetric and given by $\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2))$. One of the three possible two-particle first-excited state including both spatial and spin parts is $\Psi_{01} = [\frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2))][\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + \downarrow\uparrow\rangle)]$. The two-particle first-excited state energy is $E_{20} = 3\hbar\omega$.

Explain why you agree or disagree with each student.

****CHECKPOINT: Check your answers to questions 66a-67b. ****

66a. Ground state: $E_{00} = \hbar\omega$
First-excited state: $E_{01} = 2\hbar\omega$
66b. Ground state: $E_{01} = 2\hbar\omega$
First-excited state: $E_{02} = 3\hbar\omega$
67a. Ground state: $\Psi_{00} = \psi_0(x_1)\psi_0(x_2)$
First-excited state: $\Psi_{01} = \frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2))$
67b. Ground state: $\Psi_{01} = \frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2))$
First-excited state: $\Psi_{02} = \frac{1}{\sqrt{2}}(\psi_0(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_0(x_2))$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

8.2 One-Dimensional Harmonic Oscillator - Two Spin-1 Bosons

Two identical non-interacting spin-1 bosons ($s_1 = 1 \otimes s_2 = 1$) are placed in a one-dimensional harmonic oscillator potential energy well with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. The single-particle energies are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots$$

- For a spin-1 boson, $|s_i, m_{s_i}\rangle = \{|1,\ -1\rangle, |1,\ 0\rangle, |1,\ 1\rangle\}$ for each particle.
- When considering the spin part of the wavefunction for the two spin-1 particles ($s_1 = 1 \otimes s_2 = 1$) in the uncoupled representation in the product space, we will use the notation $|1,1\rangle_1|1,\ 1\rangle_2$, $|1,1\rangle_1|1,\ 0\rangle_2$, $|1,1\rangle_1|1,\ -1\rangle_2$, $|1,0\rangle_1|1,\ 1\rangle_2$, $|1,0\rangle_1|1,\ 0\rangle_2$, $|1,0\rangle_1|1,\ -1\rangle_2$, $|1,-1\rangle_1|1,\ 1\rangle_2$, $|1,-1\rangle_1|1,\ 0\rangle_2$, and $|1,-1\rangle_1|1,\ -1\rangle_2$ for the basis states.
- In the following table, for two identical non-interacting spin-1 bosons ($s_1 = 1 \otimes s_2 = 1$), the product states for spin degrees of freedom in the coupled representation $|s,\ m_s\rangle$ (left) are given in terms of a linear combination of product states in the uncoupled representation $|s_1,\ m_{s_1}\rangle_1|s_2,\ m_{s_2}\rangle_2$ (right) using the Clebsch-Gordon table.

Product states in Coupled Representation $\|s,\ m_s\rangle$	Written in terms of product states in Uncoupled Representation $\sum\limits_{m_{s_1}+m_{s_2}=m_s} C^{s_1,s_2,s}_{m_{s_1},m_{s_2},m_s}\|s_1,\ m_{s_1}\rangle_1\|s_2,\ m_{s_2}\rangle_2$
$\|2,\ 2\rangle$	$\|1,\ 1\rangle_1\|1,\ 1\rangle_2$
$\|2,\ 1\rangle$	$\frac{1}{\sqrt{2}}(\|1,\ 1\rangle_1\|1,\ 0\rangle_2 + \|1,\ 0\rangle_1\|1,\ 1\rangle_2)$
$\|1,\ 1\rangle$	$\frac{1}{\sqrt{2}}(\|1,\ 1\rangle_1\|1,\ 0\rangle_2 - \|1,\ 0\rangle_1\|1,\ 1\rangle_2)$
$\|2,\ 0\rangle$	$\frac{1}{\sqrt{6}}\|1,\ 1\rangle_1\|1,\ -1\rangle_2 + \sqrt{\frac{2}{3}}\|1,\ 0\rangle_1\|1,\ 0\rangle_2 + \frac{1}{\sqrt{6}}\|1,\ -1\rangle_1\|1,\ 1\rangle_2$
$\|1,\ 0\rangle$	$\frac{1}{\sqrt{2}}(\|1,\ 1\rangle_1\|1,\ -1\rangle_2 - \|1,\ -1\rangle_1\|1,\ 1\rangle_2)$
$\|0,\ 0\rangle$	$\frac{1}{\sqrt{3}}\|1,\ 1\rangle_1\|1,\ -1\rangle_2 - \frac{1}{\sqrt{3}}\|1,\ 0\rangle_1\|1,\ 0\rangle_2 + \frac{1}{\sqrt{3}}\|1,\ -1\rangle_1\|1,\ 1\rangle_2$
$\|2,\ -1\rangle$	$\frac{1}{\sqrt{2}}(\|1,\ 0\rangle_1\|1,\ -1\rangle_2 + \|1,\ -1\rangle_1\|1,\ 0\rangle_2)$
$\|1,\ -1\rangle$	$\frac{1}{\sqrt{2}}(\|1,\ 0\rangle_1\|1,\ -1\rangle_2 - \|1,\ -1\rangle_1\|1,\ 0\rangle_2)$
$\|2,\ -2\rangle$	$\|1,\ -1\rangle_1\|1,\ -1\rangle_2$

68. Find the two-particle ground state and first-excited state energies of the two-particle system if the particles are indistinguishable bosons with spin 1.

69. Construct at least two possible overall two-particle ground state wavefunctions (including both spatial and spin parts) for two non-interacting particles in the one-dimensional harmonic oscillator potential energy well if the particles are indistinguishable bosons with spin 1.

70. Construct at least two possible overall two-particle first-excited state wavefunctions (including both spatial and spin parts) for two non-interacting particles in the one-dimensional harmonic oscillator potential energy well if the particles are indistinguishable bosons with spin 1.

Consider the following conversation regarding the two-particle ground state and ground state energy for two non-interacting indistinguishable bosons with spin 1 ($s_1 = 1 \otimes s_2 = 1$) placed in a one-dimensional harmonic oscillator potential energy well.

Student 1: The two-particle ground state for a system of two indistinguishable bosons with spin 1 ($s_1 = 1 \otimes s_2 = 1$) must be symmetric. There are two possibilities for the two-particle ground state: both the spatial part and the spin part are symmetric or both the spatial part and spin part are antisymmetric.

Student 2: While that is generally the case, the two-particle ground state must be a state with the lowest energy. The lowest energy occurs when both bosons are in the same single-particle spatial state ψ_0. Therefore, the spatial part of the two-particle ground state is the symmetric state $\psi_0(x_1)\psi_0(x_2)$. The two-particle ground state energy is $E_{00} = \hbar\omega$.

Student 3: I agree with Student 2. Since the spatial part of the two-particle ground state is symmetric, the spin part of the two-particle ground state must also be symmetric. Six possible symmetric combinations for the spin part of the many-particle state for two indistinguishable spin 1 bosons ($s_1 = 1, s_2 = 1$) in the coupled representation are $|2,\ 2\rangle$, $|2,\ 1\rangle$, $|2,\ 0\rangle$, $|0,\ 0\rangle$, $|2,\ -1\rangle$, and $|2,\ -2\rangle$ in the preceding table. One possible overall two-particle ground state including both spatial and spin parts is $\Psi_{00} = [\psi_0(x_1)\psi_0(x_2)[|2,\ 2\rangle$.

Student 2: I agree with Student 3. We can also construct a completely symmetric spin state by taking a linear combination of these symmetric states. $C_1|2,\ 2\rangle + C_2|2,\ 1\rangle + C_3|2,\ 0\rangle + C_4|0,\ 0\rangle + C_5|2,\ -1\rangle + C_6|2,\ -2\rangle$ where $|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 + |C_5|^2 + |C_6|^2 = 1$ will yeild a normalized state.

Explain why you agree or disagree with each student.

Consider the following conversation regarding the two-particle first-excited state and first-excited state energy for two non-interacting indistinguishable spin 1 bosons ($s_1 = 1 \otimes s_2 = 1$) placed in a one-dimensional harmonic oscillator potential energy well.

Student 1: If the two-particle first-excited state energy is $E_{01} = 2\hbar\omega$, one boson is in the single-particle spatial state ψ_0 and the other boson is in the single-particle spatial state ψ_1. The spatial part of the two-particle first-excited state MUST be $\frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)]$ since the overall wavefunction must be symmetric. Therefore, the spin part of the two-particle first-excited state must be a symmetric spin state.

Student 2: The spatial part of the two-particle first-excited state can also be $\frac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)]$, in which case the spin part of the two-particle first-excited state must be an antisymmetric spin state.

Explain why you agree or disagree with each student.

In the preceding conversation, Student 1 is correct that both the spatial and spin part of the two-particle stationary state wavefunction can be symmetric to produce an overall symmetric first-excited state wavefunction for the two bosons. However, it is also possible that both the spatial and spin parts of the two-particle stationary state wavefunction can be antisymmetric resulting in an overall symmetric first-excited state wavefunction for the two bosons as stated by Student 2.

****CHECKPOINT: Check your answers to questions 68-69. ****

68. Ground state: $E_{00} = \hbar\omega$
First-excited state: $E_{01} = 2\hbar\omega$
69. We will use the following notation, $|s,\ m_s\rangle_i$ represents the spin state of particle i.
Ground State:

$$\begin{aligned}
\Psi_{00,1} &= \psi_0(x_1)\psi_0(x_2)][|1,\ 1\rangle_1|1,\ 1\rangle_2] \\
\Psi_{00,2} &= \psi_0(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 1\rangle_1|1,\ 0\rangle_2 + |1,\ 0\rangle_1|1,\ 1\rangle_2)] \\
\Psi_{00,3} &= \psi_0(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{6}}|1,\ 1\rangle_1|1,\ -1\rangle_2 + \sqrt{\tfrac{2}{3}}|1,\ 0\rangle_1|1,\ 0\rangle_2 + \tfrac{1}{\sqrt{6}}|1,\ -1\rangle_1|1,\ 1\rangle_2] \\
\Psi_{00,4} &= \psi_0(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{3}}|1,\ 1\rangle_1|1,\ -1\rangle_2 - \tfrac{1}{\sqrt{3}}|1,\ 0\rangle_1|1,\ 0\rangle_2 + \tfrac{1}{\sqrt{3}}|1,\ -1\rangle_1|1,\ 1\rangle_2] \\
\Psi_{00,5} &= \psi_0(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 0\rangle_1|1,\ -1\rangle_2 + |1,\ -1\rangle_1|1,\ 0\rangle_2)] \\
\Psi_{00,6} &= \psi_0(x_1)\psi_0(x_2)][|1,\ -1\rangle_1|1,\ -1\rangle_2]
\end{aligned}$$

70. We will use the following notation, $|s_i,\ m_{s_i}\rangle_i$ represents the spin state of particle i.
First-excited state:

$$\begin{aligned}
\Psi_{01,1} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)][|1,\ 1\rangle_1|1,\ 1\rangle_2] \\
\Psi_{01,2} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 1\rangle_1|1,\ 0\rangle_2 + |1,\ 0\rangle_1|1,\ 1\rangle_2)] \\
\Psi_{01,3} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{6}}|1,\ 1\rangle_1|1,\ -1\rangle_2 + \sqrt{\tfrac{2}{3}}|1,\ 0\rangle_1|1,\ 0\rangle_2 + \tfrac{1}{\sqrt{6}}|1,\ -1\rangle_1|1,\ 1\rangle_2] \\
\Psi_{01,4} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{3}}|1,\ 1\rangle_1|1,\ -1\rangle_2 - \tfrac{1}{\sqrt{3}}|1,\ 0\rangle_1|1,\ 0\rangle_2 + \tfrac{1}{\sqrt{3}}|1,\ -1\rangle_1|1,\ 1\rangle_2] \\
\Psi_{01,5} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 0\rangle_1|1,\ -1\rangle_2 + |1,\ -1\rangle_1|1,\ 0\rangle_2)] \\
\Psi_{01,6} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)][|1,\ -1\rangle_1|1,\ -1\rangle_2] \\
\Psi_{01,7} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 1\rangle_1|1,\ 0\rangle_2 - |1,\ 0\rangle_1|1,\ 1\rangle_2)] \\
\Psi_{01,8} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 1\rangle_1|1,\ -1\rangle_2 - |1,\ -1\rangle_1|1,\ 1\rangle_2)] \\
\Psi_{01,9} &= \tfrac{1}{\sqrt{2}}[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)][\tfrac{1}{\sqrt{2}}(|1,\ 0\rangle_1|1,\ -1\rangle_2 - |1,\ -1\rangle_1|1,\ 0\rangle_2)]
\end{aligned}$$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

71. Consider a system with three identical non-interacting spin-1 particles. If the three particles are in different spin states, construct a completely symmetric spin state for the three particles starting with the basis state $|1,\ 1\rangle_1|1,\ 0\rangle_2|1,\ -1\rangle_3$. If no such spin state exists, state the reason why.

72. Consider a system with three identical non-interacting spin-1 particles. If the three particles are in different spin states, construct a completely antisymmetric spin state for the three particles starting with the basis state $|1,\ 1\rangle_1|1,\ 0\rangle_2|1,\ -1\rangle_3$. If no such spin state exists, state the reason why.

****CHECKPOINT: Check your answers to questions 71-72. ****

71. $\frac{1}{\sqrt{6}}[|1,\ 1\rangle_1|1,\ 0\rangle_2|1,\ -1\rangle_3+|1,\ 1\rangle_1|1,\ 0\rangle_3|1,\ -1\rangle_2+|1,\ 1\rangle_2|1,\ 0\rangle_3|1,\ -1\rangle_1+|1,\ 1\rangle_2|1,\ 0\rangle_1|1,\ -1\rangle_3+|1,\ 1\rangle_3|1,\ 0\rangle_1|1,\ -1\rangle_2+|1,\ 1\rangle_3|1,\ 0\rangle_2|1,\ -1\rangle_1]$

72. $\frac{1}{\sqrt{6}}[|1,\ 1\rangle_1|1,\ 0\rangle_2|1,\ -1\rangle_3-|1,\ 1\rangle_1|1,\ 0\rangle_3|1,\ -1\rangle_2+|1,\ 1\rangle_2|1,\ 0\rangle_3|1,\ -1\rangle_1-|1,\ 1\rangle_2|1,\ 0\rangle_1|1,\ -1\rangle_3+|1,\ 1\rangle_3|1,\ 0\rangle_1|1,\ -1\rangle_2-|1,\ 1\rangle_3|1,\ 0\rangle_2|1,\ -1\rangle_1]$

If any of your answers do not match the checkpoint, go back and reconcile any differences you may have with the checkpoint answer.

OPTIONAL: This final optional section of this tutorial deals with examples of limiting cases when identical paticles can be treated as distinguishable.

9 Limiting Case: When Identical Particles Can Be Treated as Distinguishable

- So far we considered the distinguishable particle case as a hypothetical case for contrast with the cases of identical fermions and identical bosons. Now we will learn about some limiting cases in which identical microscopic particles can be treated as distinguishable.
- In limiting situations in which identical particles (particles of one type with the same properties) can be treated as distinguishable, you can distinguish which particle is in which single-particle stationary state. Exchanging distinguishable particles in different single-particle states with each other produces a distinctly different many-particle state.

Consider the following conversation regarding identical particles which can be treated as distinguishable versus indistinguishable.
Student 1: In general, in quantum mechanics, if two particles in a system are identical, we couldn't paint one red and the other green. Quantum particles are truly indistinguishable. There is no measurement we can perform that could distinguish one identical particle from the other. For example, there is no measurement that can distinguish which fermion was in which single-particle state and had which coordinate.
Student 2: I agree with Student 1. Identical particles are indistinguishable. However, under certain circumstances, for example, when the overlap of the single-particle wavefunctions is negligible, we can treat the particles as distinguishable.

Explain why you agree or disagree with the students.

Consider the following conversation regarding when identical particles (particles of the same type with the same properties) can be treated as distinguishable.
Student 1: In nature, aren't all identical microscopic particles with the same properties, e.g., electrons, indistinguishable? How can we consider the identical particles as distinguishable?
Student 2: That is a good question! In certain limits, microscopic identical particles can be treated as distinguishable. For example, when the overlap of the single-particle wavefunctions of the identical fermions or identical bosons is negligible, we can treat them as distinguishable particles. As an example, if we are considering electrons in two metal blocks with a macroscopic separation between them, then there is negligible overlap in their single-particle wavefunctions and the electrons in the two metal blocks can be treated as distinguishable from those in the other block.
Student 3: I agree with Student 2. Also, in the classical limit, for a system of electrons at "high" temperature, the de Broglie wavelength of the electron in a material becomes small compared to the average separation between the particles. The overlap of the single-particle wavefunctions for the electrons becomes negligible and the electrons can be treated as distinguishable.

Explain why you agree or disagree with Student 2 and Student 3.

73. Consider a system of two non-interacting, identical particles in the limiting case in which they can be treated as distinguishable. $\psi_{n_1}(x)$ and $\psi_{n_2}(x)$ are the single-particle wavefunctions for the system ($n_1 \neq n_2$). Choose all of the following wavefunctions that are appropriate two-particle stationary state wavefunctions for a system of two non-interacting, identical particles if they can be treated as distinguishable.

(a) $\psi_{n_1}(x_1)\psi_{n_2}(x_1)$ (same label x_1)

(b) $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$

(c) $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$ (same label n_1)

(d) $\psi_{n_1}(x)\psi_{n_1}(x)$

Consider the following conversation regarding appropriate wavefunctions for a system of two non-interacting identical particles in the limiting case in which they can be treated as distinguishable.
Student 1: For a system of two non-interacting identical particles which can be treated as distinguishable, the wavefunction describing the system can be $\psi_{n_1}(x_1)\psi_{n_2}(x_2)$. Here $\psi_{n_1}(x_1)$ means that particle 1 with coordinate x_1 is in a single-particle state denoted by n_1. Similarly, $\psi_{n_2}(x_2)$ means that particle 2 with coordinate x_2 is in a single-particle state denoted by n_2.
Student 2: I agree with Student 1. In this limiting case, we can treat the identical particles independently and we can just multiply their single-particle wavefunctions. There is no need to symmetrize or antisymmetrize the many-particle stationary state wavefunction.

Explain why you agree or disagree with the students.

Consider the following conversation regarding the appropriate wavefunction for a system of two non-interacting identical fermions which can be treated as distinguishable.
Student 1: For a system of two non-interacting identical fermions which can be treated as distinguishable, it is possible for the wavefunction describing the system to be $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$. Here $\psi_{n_1}(x_1)$ means that particle 1 with coordinate x_1 is in a single-particle state denoted by n_1. Similarly, $\psi_{n_1}(x_2)$ means that particle 2 with coordinate x_2 is in a single-particle state denoted by n_1.
Student 2: I disagree with Student 1. Two fermions can never be in the same single-particle state even in limiting cases for which fermions can be treated as distinguishable.
Student 3: I agree with Student 2. In limiting cases where fermions can be treated as distinguishable, the average occupancy of each single-particle state is less than 1. In this case, we can treat the fermions independently and we can just multiply their single-particle wavefunctions in which all the single-particle states have different indices. There is no need to antisymmetrize the many-particle stationary state wavefunction.

Explain why you agree or disagree with the students.

Consider the following conversation regarding the appropriate wavefunction for a system of two non-interacting <u>identical bosons</u> which can be treated as distinguishable.

Student 1: For a system of two non-interacting identical bosons which can be treated as distinguishable, the wavefunction describing the system can be $\psi_{n_1}(x_1)\psi_{n_1}(x_2)$. $\psi_{n_1}(x_1)$ means that particle 1 with coordinate x_1 is in a single-particle state denoted by n_1. Similarly, $\psi_{n_1}(x_2)$ means that particle 2 with coordinate x_2 is in a single-particle state denoted by n_1.

Student 2: I agree with Student 1. There is nothing that prohibits two bosons from occupying the same single-particle state. In the limiting case in which identical bosons can be treated as distinguishable, the stationary state wavefunction is the product of the single-particle stationary state wavefunctions.

Student 3: While I agree with Student 2 that nothing forbids two identical bosons from occupying the same single-particle state, in the limit in which identical bosons can be treated as distinguishable, the average number of bosons in any given single-particle state is less than 1.

Student 4: I agree with Student 3. In this limiting case, we can just multiply their single-particle wavefunctions in which all the single-particle states have different indices. There is no need to symmetrize the many-particle stationary state wavefunction.

Explain why you agree or disagree with each students.

Consider the following conversation regarding a physical system in which two non-interacting identical bosons can be treated as distinguishable.

Student 1: If we consider two He-4 atoms separated by a distance greater than the de Broglie wavelength such that there is negligible overlap in their single-particle wavefunctions, we can treat the He-4 atoms as distinguishable and treat each atom independently.

Student 2: I agree. For example, if we treat each He-4 atom as a separate system and each is in its OWN ground state, the two-particle stationary state wavefunction would be the product of the single-particle ground state wavefunctions for each He-4 atom.

Student 3: I disagree with Student 2. If both He-4 atoms are in their ground states, then the He-4 atoms are in the same single-particle state ψ_1. The two-particle stationary state wavefunction would be $\Psi(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$.

Student 2: I disagree with Student 3. Even though the He-4 atoms are both in their respective ground states, the He-4 atoms are not in the SAME single-particle state because they are separated spatially by a macroscopic distance. They are essentially two different systems. There is no overlap in these ground state wavefunctions for the two He-4 atoms.

Student 1: I agree with Student 2. Perhaps using identifiers for the two ground states would help. For example, the two-particle stationary state wavefunction would be $\Psi(x_1, x_2) = \psi_1(x_1)\psi_1'(x_2)$ in which $\psi_1(x_1)$ is the ground state of the first He-4 atom and $\psi_1'(x_2)$ is the ground sate of the second He-4 atom.

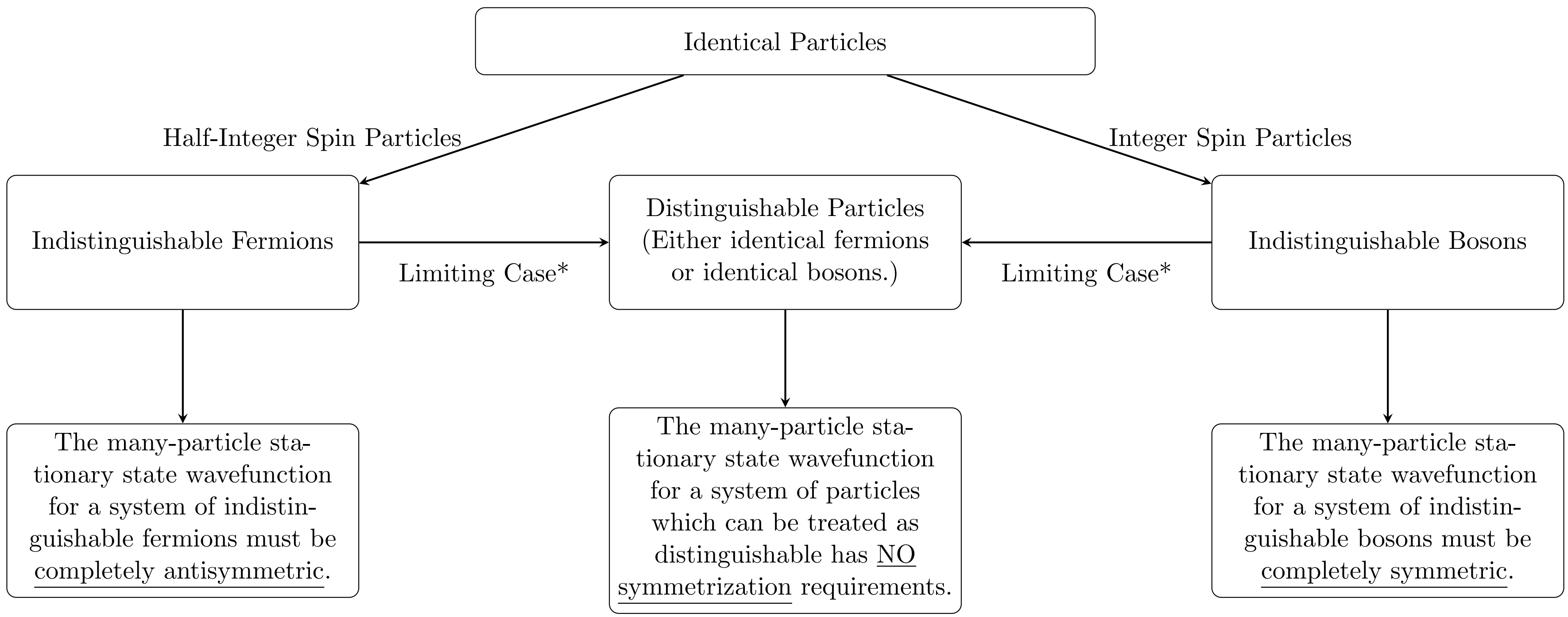
Explain why you agree or disagree with each students.

****CHECKPOINT: Check your answer to question 73. ****

73. b

If your answer does not match the checkpoint, go back and reconcile any difference you may have with the checkpoint answer.

Review the following flowchart which summarizes the properties of non-interacting identical particles



* In certain circumstances, e.g., when the overlap of the wavefunctions of the identical particles is negligible, we can treat them as distinguishable. In this limiting case, the average occupancy of each single-particle state is less than 1 and Pauli's exclusion principle is not violated.