**Understanding the Mach-Zehnder Interferometer (MZI) with Single Photons: Homework**

**The goals of this homework are to use a simplified ideal version of MZI to help you:**

* **Connect qualitative understanding of the MZI with a simple mathematical model**
* **After choosing a basis, you will:**
1. **Determine the matrix representation of quantum mechanical operators that correspond to beam splitter 1 and beam splitter 2**
2. **Determine the matrix representation of the quantum mechanical operator that corresponds to the mirrors**
3. **Find the action of the beam splitters BS1 and BS2 on an input state and the probability of a detector D1or D2 “clicking”**
4. **Determine the matrix representation of a quantum mechanical operator that corresponds to a phase shifter (e.g., a glass piece of a certain thickness inserted into one of the paths of the MZI)**
5. **Find probabilities of detectors “clicking” after a phase shifter is inserted and acquire intermediate interference at detectors D1 and D2 (neither fully constructive nor fully destructive)**
6. **Verify that the matrix representation of the quantum mechanical operators corresponding to beam splitter 1, beam splitter 2, mirrors, phase shifter, and node evolve the state of the system and are unitary operators which preserve the norm of the physical state**

The setup for the ideal Mach-Zehnder interferometer (MZI) shown below is as follows:

* All angles of incidence are 45° with respect to the normal to the surface.
* For simplicity, we will assume that a photon can only reflect from one of the two surfaces of the identical half-silvered mirrors (beam splitters) BS1 and BS2 because of anti-reflection coatings.
* The detectors D1 and D2 are point detectors located symmetrically with respect to the other components of the MZI as shown.
* The photons originate from a monochromatic coherent point source. (Note: Experimentally, a source can only emit nearly monochromatic photons such that there is a very small range of wavelengths coming from the source. Here, we assume that the photons have negligible “spread” in energy.)
* Assume that the photons propagating through both the U and L paths travel the same distance in vacuum to reach each detector.
* In the classical usage of the Mach-Zehnder Interferometer, a “beam” of light is sent which gets separated spatially after passing through BS1. Now, we will send one photon at a time through the MZI.
* In all of the discussions below, ignore the effect of polarization of the photons due to reflection by the beam splitters or mirrors.
* Assume that the photons coming from the single photon source are unpolarized.
* Assume that beam splitters BS1 and BS2are infinitesimally thin so that there is no phase shift when a photon propagates through them.
* For the entire tutorial, assume that a large number (N) of photons are sent one at a time.



Before we begin, we will make a few assumptions:

* The beam splitters are 50/50 splitters, meaning that a measurement of the photon position immediately after it exits the beam splitter BS1 would yield an outcome such that the photon is either in the upper path or the lower path with 50% probability.
* The silvered side of the beam splitter is the point of reflection. No reflection occurs at the air-glass interface (the bold side of the beam splitter), due to anti-reflection coatings.
* From here on, assume that the thickness of the beam splitters is negligible so the phase shift introduced by the propagation of light through the beam splitters is zero ().
* No relative phase shift is introduced when a photon propagates through vacuum because the photon travels the same distance in vacuum along each of the U and L paths.
* The upper path is marked in RED. The lower path is marked in BLACK.

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1. **Determining the matrix representation of the quantum mechanical operators that correspond to beam splitter 1 (BS1) and beam splitter 2 (BS2)**

In a simplified quantum mechanical model of a photon state which accounts for the two paths and , a single photon traveling through the MZI can be considered to be a two state quantum system.



 Figure 1.a : photon initially in state Figure 1.b: photon initially in state

* Let’s consider **BS1**. Note that in Figure 1.a and Figure 1.b, depending on the position of the source, a photon can travel along different paths before it enters BS1. We will call these initial paths (upper path) and (lower path).
* Let’s choose a basis in which we denote the state of the photon from the source towards BS1 along the upper path by as shown in Figure 1.a and denote the state along the lower path by as shown in Figure 1.b.
* After exiting BS1, the quantum mechanical state of the photon is in an equal superposition of the upper and lower states (since BS1 is a 50/50 beam splitter).
* If a measurement of the photon’s position is made immediately after BS1 by a photo-detector, the measurement would yield the photon either in the RED path, denoted by upper state , or the BLACK path, denoted by lower state .

Here is a summary of the phase shifts from the MZI warm-up. Use this table to find the phase shifts for the photon that enters BS1 and goes into a state which is an equal superposition of the and states. Knowing the phase shifts will help us determine the superposition state of the photon after exiting BS1 and the operator matrix for BS1.

Table 1

|  |  |  |
| --- | --- | --- |
|  | Initially in medium with lower *n* | Initially in medium with higher *n* |
| Reflection at interface | Phase shift of π | No phase shift |
| Transmission at interface | No phase shift | No phase shift |
| Propagation through a medium | Phase shift φ depends on thickness and refractive index *n* |

* Since there are two linearly independent states of the photon in this simplified model corresponding to the two paths and , the Hilbert space is two dimensional.
* Any operator in a two dimensional Hilbert space can be represented by a matrix in the chosen basis.
* We can think of the beam splitters and mirrors as quantum mechanical operators which act on the photon state and can change it at various times during its propagation through the MZI.

The **BS1** operator acts on a photon in an initial state (either or , depending on the location of the source) and puts it into an equal superposition of the upper and lower states. We will denote the quantum mechanical operator corresponding to BS1 as [BS1].



Figure 1.a: photon initially in state

1. Consider the following statements from three students about the effect of operator [BS1] on the initial photon state shown in Figure 1.a
* Student 1: [BS1]= since the operator [BS1] puts the photon state into an equal superposition of and .
* Student 2: No, [BS1]= because there is a phase shift of π for the part that passes through BS1 and .
* Student 3: No. [BS1]= since there is a phase shift of π for the part that reflects and no phase shift for the part that passes through BS1.

Which students, if any, do you agree with and why? (Hint: look at Table 1)



Figure 1.a: photon initially in state

The action of the operator [BS1] on the photon in the initial state as shown in Figure 1.a should be

[BS1]()=[BS1]= (1)

The ensures that after the photon exits BS1, there is a 50% probability that a measurement of the photon’s position would yield the photon either in the upper or lower state. Using Table 1, the upper state is multiplied by -1 because of the reflection at the vacuum-BS1 interface (initially in a medium with a lower index of refraction). Thus, the upper state undergoes a phase change of π. Therefore, we multiply the upper state by . The lower state is transmitted at the vacuum-BS1 interface, so it is not phase shifted (see Table 1)

1. Call the [BS1] matrix in the given basis . Using equation (1) and , determine the matrix elements and .

Now let’s determine matrix elements and of the operator [BS1].



Figure 1.b: photon initially in state

1. Consider Figure 1.b so that the initial state of the photon before passing through BS1 is . Use Table 1 to find the phase shifts of the upper and lower states immediately after the photon exits BS1.
2. The action of the operator BS1 on the photon in the initial state as shown in Figure 1.b should be

[BS1]([BS1]= (2)

Since the upper and lower states are not phase shifted in this case (see Table 1), there are no minus signs. Using equation (2) and , find the matrix elements and of the operator [BS1].

1. Write down the matrix for the operator [BS1] corresponding to beam splitter 1 in terms of what you found in the preceding questions 1-4.

Now we need to find the matrix representation of the operator[BS2], the quantum mechanical operator corresponding to beam splitter **BS2** for the two state system for a photon we are considering corresponding to the two possible paths and . (Note: The matrix representation of [BS2] we obtain will correspond to our convention in which the path leading to D2 is and the path leading to D1 is .)





 Figure 2.a Figure 2.b

1. In Figure 2.a, the photon enters BS2 from the upper path. Use Table 1 to find the phase shifts of the upper and lower states after the photon exits BS2.
2. Using the phase shifts identified in the preceding question, what is [BS2] ?
3. [BS2]=
4. [BS2]=
5. [BS2]=
6. [BS2]=
7. The correct answer for question 7 is (d), since neither the upper state nor lower state is phase shifted. Let the operator matrix in the given basis be [BS2]= . Find the matrix elements and using the answer to the preceding question.



Figure 2.b

1. In Figure 2.b, the photon enters BS2 from the lower path. Use Table 1 to find the phase shifts of the upper and lower states after the photon exits BS2.
2. Using the phase shifts identified in the preceding question, what is [BS2] ?
3. [BS2]=
4. [BS2]=
5. [BS2]=
6. [BS2]=
7. The correct answer to the preceding question 10 is (b), since the lower state undergoes a phase change of π. Therefore, we multiply the lower state by . The upper state is not phase shifted. Use your answers to the preceding questions to find the matrix elements and of the [BS2] matrix.
8. Write down the matrix for the operator [BS2] corresponding to beam splitter 2 in terms of what you found in the preceding questions.
9. **Determining the matrix representation of the quantum mechanical operators that correspond to the mirrors (M)**

Let’s now consider the action of the **mirrors** on the photon states.



Figure 3

* In Figure 3, mirror 1 (M1) reflects only the photon state in the lower path and mirror 2 (M2) reflects only the photon state in the upper path. Thus, we can think of the mirrors as quantum mechanical reflection operators.
* The phase shifts of the photon state in the lower and upper paths due to reflection off the mirrors is π, and . The mirror operator is a reflection operator that should multiply the incoming state by a factor and act on both the upper and lower paths.
* We will denote the quantum mechanical operator corresponding to mirror 1 as [M1] and the quantum mechanical operator corresponding to mirror 2 as [M2].
* Since M1 only phase shifts the photon state in the lower path by , [M1]=.
* Since M2 only phase shifts the photon state in the upper path by , [M2]=.
1. Consider the following conversation between Student A and Student B:
* Student A: Suppose the initial state of the photon is the upper state, e.g., . After the photon propagates through BS1, the photon state is given by equation (1) which is

[BS1]()=[BS1]=

To check that the mirror operator [M1] only shifts the photon state in the lower path , we can write [M1]. So we see that has been phase shifted by π and multiplied by

* Student B: I agree. To check that the mirror operator [M2] only shifts the photon state in the upper path , we can write

 [M2]. So we see that has been phase shifted by π and multiplied by

Do you agree with Student A and Student B’s reasoning? Explain.

1. Mirror 1 and Mirror 2 produce reflection at the same time, e.g., , on the photon state in the lower and upper paths, respectively. Therefore, the net effect of the mirrors on the state of the photon at time is the product of the two operators [M1][M2]. Combine the effects of the two mirrors on the photon state and find the matrix representation for the operator [M]=[M1][M2] that corresponds to the action of M1 and M2 on the photon state.
2. **Finding the action of the beam splitters and mirrors on an input state and their effect on the probability of a detector D1 or D2 “clicking” after a photon is emitted from the source**

We now have the matrix representations of the beam splitters and mirrors:

[BS1]=

[BS2]=

[M]=, where is the identity operator.

1. Find the action of these optical elements on a photon in an initial state as shown in Figure 3. [BS2][M][BS1]()=?

Figure 3

1. Describe the implication of the answer to your preceding question on obtaining constructive or destructive interference at detectors D1 and D2 for the case in which the photons started in state .
2. Consider the following conversation between two students:
* Student 1: [BS2][M][BS1]=. So [BS2][M][BS1]()=. This implies that if the initial state of the photon is =, the photon will end up in detector D1 since that is the path and corresponds to =.
* Student 2. I agree with you. And [BS2][M][BS1]()=, which implies that if the initial state of the photon is =, the photon will end up in detector D2 since that is the path and corresponds to =. The expressions for the operators [BS1] and [BS2] we came up with were based upon how we defined the and paths.

Do you agree with Student 1 and Student 2? Explain your reasoning.

1. In the MZI warm-up, you found that the path lengths of the upper and lower paths were such that at detector D1 there was constructive interference and at detector D2 there was destructive interference for a photon that was initially in the state =. Does your answer for the preceding question (question 16) agree with this earlier finding? Explain your reasoning.
2. Suppose the source is set up as in Figure 1.b such that the initial state of the photon is = . Find the action of [BS2][M][BS1]()=?
3. Does your answer to the preceding question agree with Student 2’s statement in question 17? If not, go back and check your work.
4. Describe the implication of the answer to the preceding question in obtaining constructive or destructive interference at detectors D1 and D2. Explain whether for the case in which the photons started in the state = you will see constructive interference at detector D1 or D2.
5. **Determine the matrix representation of a quantum mechanical operator that corresponds to a phase shifter (PS) (e.g., a glass piece of a certain thickness inserted into one of the paths of MZI)**

Suppose we now insert a phase shifter (piece of glass with a certain thickness) in the upper path as shown in Figure 4 which shifts the photon state by a phase .

Figure 4

Let’s find the matrix representation of operator corresponding to the phase shifter.

* Recall that after the photon exits BS1, the photon state is
* As shown in Figure 4, after the photon exits BS1, the next optical element to act on the photon state is the phase shifter (glass piece). Since the phase shifter is an operator which only acts on the upper state, the matrix representation of the phase shifter [PS] must shift the upper state like this: , but should not change the lower state
* The only matrix that will accomplish this operation is [PS]=.

Let’s check that the phase shifter operator only shifts the upper state by and leaves the lower state unchanged, e.g,

[PS]

Note: can be calculated if you are given the thickness of the phase shifter and the index of refraction, but in this homework will be stated explicitly in each case.

1. What would be the matrix representation of the phase shifter if it was placed in the lower path (somewhere between BS1 and BS2) which shifts the photon state by a phase in the lower path ?
2. **Finding probabilities of detectors “clicking” after a phase shifter is inserted due to intermediate interference at the detectors**

For the following questions 23-26, assume Figure 4 is the setup of the MZI.



Figure 4

1. Let’s suppose we know the phase angle of the phase shifter is . What is the final state of the photon that starts in state = when it exits BS2? Hint: Use the matrix representations of the operators [BS1], [PS], [M], and [BS2] and act with these operators on the state = in the appropriate time-ordered manner.
2. Based upon your response to the preceding question, what is the probability that detector 1 clicks?
3. Based upon your response to question 23, what is the probability that detector 2 clicks?
4. In the MZI warm-up, you found that inserting a phase shifter will create intermediate interference (neither completely constructive nor destructive) at both detectors with different probabilities. Do your answers to the preceding two questions agree with this? Explain your reasoning.

**Removable BS2**

* In the MZI tutorial, you learned that when BS2 is removed:
	+ You have “which-path” information about the photon when the photon arrives at detectors D1 or D2.
	+ In particular, if detector D1 clicks, the photon must have come from the path and if detector D2 clicks, the photon must have come from the path (see Figure 5). There will no longer be interference observed at detectors D1 or D2 and the probability of each detector clicking is 50% , regardless of the relative phase difference of the two paths(because the photon will not interfere with itself if we have “which-path” information).



Figure 5

Let’s check this mathematically. Suppose the photon state is initially =.

* In the MZI tutorial, you learned that, without BS2, if detector D1 clicks, the photon must have come from the path and if detector D2 clicks, the photon must have come for the path.
* However, we arbitrarily defined the upper path as RED and lower path as BLACK and used this convention to come up with the matrix representation of the operator [BS2].
* Therefore, without BS2, if detector D1 clicks, the photon came from path and if detector D2 clicks, the photon must have come from the path.
* Therefore, in Figure 5 at the junction before detectors D1 and D2, we must define a node operator, [NODE], to exchange the and states when BS2 is not present. The matrix for the operator in the chosen basis (=, =) that will accomplish this operation is [NODE]=.

The final state of the photon entering BS1 in state = after propagating through BS1, reflecting off the mirrors, and exiting the node is

[NODE][M][BS1]=

1. Based upon the preceding discussion of the final state of the photon without BS2, what is the probability that detector D1 and detector D2 will click for the case in Figure 5?



Figure 6

1. Suppose we now insert a glass piece in the upper path of the MZI.
2. If the initial state of the photon is =, what is the final state of the photon after it passes through the node but before the state collapses due to interaction with the detectors D1 and D2?
3. Based upon your response to the preceding question, what is the probability that detectors D1 and D2 will click?
4. Does inserting a phase shifter (see Figure 6) into the or path of the MZI affect the probabilities of detectors D1 and D2 clicking for the case without BS2 for which we can obtain “which-path” information?
5. Based upon your response to the preceding question, is there any interference observed at detectors D1 and D2 when BS2 is not present, as in Figure 6?
6. Explain why there is no interference at detectors D1 and D2 without BS2.
7. A) Consider the following conversation between three students:
* Student A: The operators for BS1, BS2, mirrors, phase shifter, and node that we found are hermitian operators and correspond to physical observables.
* Student B: I disagree with you. The operators for BS1, BS2, mirrors, phase shifter, and node that we found are unitary operators and help us find how the state evolves in time at different instants.
* Student C: I think these operators (BS1, BS2, mirrors, phase shifter, and node) that we found are both hermitian and unitary.

Explain why you agree or disagree with each student.

1. B) If an operator is hermitian, then . Which of the operators (BS1, BS2, mirror, phase shifter, and node) are hermitian operators? Do these operators correspond to physical observables (a property that can be measured, e.g., position, momentum, or energy)? If an operator corresponds to a physical observable, must that operator be hermitian?
2. C) If an operator is unitary, then , where is the identity operator. Unitary operators preserve the norm, or inner product, of the physical space. All time evolution operators must be unitary operators. Which of the operators (BS1, BS2, mirror, phase shifter, and node) are unitary operators?
3. **Challenge problem**: We now define the upper and lower paths as shown in Figure 7 in which the path leading to detector D1 is the path and the path leading to the detector D2 is the path:



Figure 7

1. Re-calculate the matrix representation of the operator[BS2], the quantum mechanical operator corresponding to beam splitter BS2 for the two state system for a photon corresponding to the two possible paths and shown in Figure 7 above. In this case, the node operator [NODE] will turn out to be the identity operator but the matrix corresponding to the operator [BS2] will differ from the one we found in question 12.
2. For the convention chosen in Figure 7 for the and paths, check that the final state of a photon that was initially in state = is the same as the final state that you found in question 15.
3. If BS2 is removed from the setup shown in Figure 7, what is the final state of a photon that was initially in state = using the convention for the and paths shown in Figure 7 if you define the node operator to be the identity operator? Is your answer using this new convention the same as the final state of the photon entering BS1 in state = after propagating through BS1, reflecting off the mirrors, and exiting the node that you found earlier which is

[NODE][M][BS1]=?

1. Using your answer to the preceding question, explain why the final state of the photon should be the same, regardless of the way you define the upper and lower paths.