Warm-up: Pre-test

Notation and Conventions: For a spin-one-half (spin-1/2) system, the spin quantum number $s = 1/2$ and the quantum number for the z component of the spin is $m_z = \pm 1/2$. Therefore, if we choose the orthonormal basis vectors to be the eigenstates of the operators $\hat{S}^2$ and $\hat{S}_z$, we can denote them as $|s, m_z\rangle$: $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$.

• Since the spin quantum number $s = 1/2$ is fixed, we can simplify the notation for the basis vectors as $|s, m_z\rangle \equiv |m_z\rangle \equiv |\pm 1/2\rangle$ where $m_z = \pm 1/2$.

• Another common notation for the basis vectors $|s, m_z\rangle$ for a spin-1/2 system is $|1/2\rangle \equiv |\uparrow\rangle$ and $|-1/2\rangle \equiv |\downarrow\rangle$.

• The eigenvalue equations for the operators $\hat{S}_z$ and $\hat{S}^2$ are given by:

$\hat{S}_z|\uparrow\rangle = (\hbar/2)|\uparrow\rangle$, and $\hat{S}_z|\downarrow\rangle = (-\hbar/2)|\downarrow\rangle$

$\hat{S}^2|\uparrow\rangle = \hbar^2 s(s+1)|\uparrow\rangle = (\hbar^2/2)(1/2+1)|\uparrow\rangle = (3\hbar^2/4)|\uparrow\rangle$,

and $\hat{S}^2|\downarrow\rangle = \hbar^2 s(s+1)|\downarrow\rangle = (\hbar^2/2)(1/2+1)|\downarrow\rangle = (3\hbar^2/4)|\downarrow\rangle$

In the algebra of angular momentum, the raising and lowering operators have the following effect on the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$:

$\hat{S}_\pm|sm_z\rangle = \hbar\sqrt{s(s+1) - m_z(m_z \pm 1)}|s(m_z \pm 1)\rangle$

The raising operator acting on the highest m_z state annilates it (prefactor $\sqrt{s(s+1) - m_z(m_z+1)} = 0$).

The lowering operator acting on the lowest m_z state annilates it (prefactor $\sqrt{s(s+1) - m_z(m_z-1)} = 0$).

The matrix representations for $|\uparrow\rangle$ and $|\downarrow\rangle$ are given by $|\uparrow\rangle \doteq \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and $|\downarrow\rangle \doteq \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

1. Choosing basis vectors for a vector space is equivalent to choosing
   (a) operators
   (b) eigenvalues
   (c) a coordinate system
   (d) a Hilbert space

2. Choose all of the following statements that are true about the Hamiltonian $\hat{H}_0 = \text{constant}\ \hat{S}_z$ for a spin-1/2 system:

   (I) If we choose two different bases (coordinates), $\hat{H}_0$ may be a diagonal matrix in one basis but not in the other.
   (II) If the basis vectors are eigenstates of $\hat{H}_0$, $\hat{H}_0$ will be a diagonal matrix.
   (III) No matter what basis we choose, $\hat{H}_0$ must always be a diagonal matrix by definition.

   (a) (I) only
   (b) (II) only
   (c) (I) and (II) only.
   (d) (II) and (III) only

3. Which one of the following is a matrix representation of $\hat{S}^2$ if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix.
   (a) $(3\hbar^2/4)\left(\begin{smallmatrix}1 & 0\\0 & -1\end{smallmatrix}\right)$
   (b) $(3\hbar^2/4)\left(\begin{smallmatrix}0 & 1\\1 & 0\end{smallmatrix}\right)$
   (c) $(3\hbar^2/4)\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)$
   (d) $(3\hbar^2/4)\left(\begin{smallmatrix}0 & -1\\1 & 0\end{smallmatrix}\right)$

4. Which one of the following is the correct expression for $\langle\downarrow|\hat{S}_-|\uparrow\rangle$ for a spin-1/2 system?
   (a) 0
   (b) $\hbar$
   (c) $\hbar/2$
   (d) $-\hbar$

5. Which one of the following is the correct expression for the matrix representation of the lowering operator $\hat{S}_-$ of a spin-1/2 system if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix?
   (Hint: Use the matrix element you calculated.)
   (a) $\hbar\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$
   (b) $\hbar\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$
   (c) $\hbar\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$
   (d) $\hbar\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$

Warm-up: Basics of single spin-one-half (spin-1/2) system

The goals of this tutorial are to review the basics of

- a single spin system
- dimensionality, basis vectors and matrix form of operators in the chosen basis for a finite dimensional vector space (e.g., vector space for a single spin system)

1. Consider a single spin-1/2 system (*e.g.*, an electron spin). What is the dimensionality of the vector space associated with the spin degrees of freedom?
   (a) 2-dimensional
   (b) 4-dimensional
   (c) finite dimensional but can be anything
   (d) infinite dimensional

2. How many linearly independent vectors (basis vectors) do you need to represent any vector in a 2-dimensional vector space?
   (a) 1
   (b) 2
   (c) $2 \times 2 = 4$
   (d) infinite

3. Choosing basis vectors for a vector space is equivalent to choosing
   (a) operators
   (b) eigenvalues
   (c) a coordinate system
   (d) a Hilbert space

4. Consider the following statements about an n-dimensional vector space in which the state vector of a quantum system lies. Choose all of the statements that are true:

   (I) For every physical observable, there is a corresponding Hermitian operator in the vector space.

   (II) Once you have chosen a basis, you can represent any operator as an $n \times n$ matrix.

   (III) Once you have chosen a basis, you can represent any "ket" vector as a $1 \times n$ column matrix.

   (a) (I) and (II) only
   (b) (I) and (III) only
   (c) (II) and (III) only
   (d) (I), (II) and (III)

5. Choose all of the following statements that are true:

   (I) The basis vectors are not unique and can be chosen based upon convenience.

   (II) The basis vectors can be chosen to be an eigenstate of an operator corresponding to a physical observable.

   (III) The basis vectors which are eigenstates of an operator can be labeled by the quantum numbers which specify the corresponding eigenvalues.

   (a) (I) and (II) only
   (b) (I) and (III) only
   (c) (II) and (III) only
   (d) (I), (II) and (III)

Notation and Conventions: For a spin-1/2 system, the spin quantum number $s = 1/2$ and the quantum number for the z component of the spin is $m_z = \pm 1/2$. Therefore, if we choose the orthonormal basis vectors to be the eigenstates of the operators $\hat{S}^2$ and $\hat{S}_z$, we can denote them as $|s, m_z\rangle$: $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$.

• Since the spin quantum number $s = 1/2$ is fixed, we can simplify the notation for the basis vectors as $|s, m_z\rangle \equiv |m_z\rangle \equiv | \pm 1/2\rangle$ where $m_z = \pm 1/2$.

• Another common notation for the basis vectors $|s, m_z\rangle$ for a spin-1/2 system is $|1/2\rangle \equiv |\uparrow\rangle$ and $|-1/2\rangle \equiv |\downarrow\rangle$.

• The eigenvalue equations for the operators $\hat{S}_z$ and $\hat{S}^2$ are given by:

$\hat{S}_z|\uparrow\rangle = (\hbar/2)|\uparrow\rangle$, and $\hat{S}_z|\downarrow\rangle = (-\hbar/2)|\downarrow\rangle$

$\hat{S}^2|\uparrow\rangle = \hbar^2 s(s+1)|\uparrow\rangle = (\hbar^2/2)(1/2+1)|\uparrow\rangle = (3\hbar^2/4)|\uparrow\rangle$,

and $\hat{S}^2|\downarrow\rangle = \hbar^2 s(s+1)|\downarrow\rangle = (\hbar^2/2)(1/2+1)|\downarrow\rangle = (3\hbar^2/4)|\downarrow\rangle$

The matrix representations for $|\uparrow\rangle$ and $|\downarrow\rangle$, which are the eigenstates of $\hat{S}_z$, are $\binom{1}{0}$ and $\binom{0}{1}$, respectively.

6. Choose all of the following statements that are true:

(I) For a spin-1/2 system, the spin quantum number is $s = 1/2$ and the quantum number for the z component of the spin is $m_z = \pm 1/2$.

(II) The spin quantum number s is related to the eigenvalue of the "magnitude of the square of the angular momentum" operator $\hat{S}^2$, i.e., $\hat{S}^2|s, m_z\rangle = \hbar^2 s(s+1)|s, m_z\rangle$.

(III) The quantum number m_z is related to the eigenvalue of the z component of spin $\hat{S}_z$, i.e., $\hat{S}_z|s, m_z\rangle = m_z\hbar|s, m_z\rangle$.

(a) (I) and (II) only

(b) (I) and (III) only

(c) (II) and (III) only

(d) (I), (II) and (III)

7. Choose all of the following statements that are true:

(I) $|\uparrow\rangle$ and $|\downarrow\rangle$ are always the eigenstates of the given Hamiltonian.

(II) $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the operators $\hat{S}^2$ and $\hat{S}_z$.

(III) $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of $\hat{S}^2$ and $\hat{\vec{S}}$ operators.

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only.

8. Consider the following statements about the notation you learned for a spin-1/2 system. Then, choose all of the following statements that are true.

$$\text{(I) } |1/2, 1/2\rangle \equiv |1/2\rangle \equiv |\uparrow\rangle$$
$$\text{(II) } |-1/2, -1/2\rangle \equiv |-1/2\rangle \equiv |\downarrow\rangle$$
$$\text{(III) } \hat{S}_z|\downarrow\rangle = \hbar/2|\downarrow\rangle$$

(a) (I) only

(b) (II) only

(c) (I) and (II) only

(d) (I), (II) and (III)

9. Which one of the following sets of scalar (inner) products is correct?

(a) $\langle\uparrow|\uparrow\rangle = 1, \langle\uparrow|\downarrow\rangle = 1, \langle\downarrow|\uparrow\rangle = 1, \langle\downarrow|\downarrow\rangle = 1.$

(b) $\langle\uparrow|\uparrow\rangle = 1, \langle\uparrow|\downarrow\rangle = 0, \langle\downarrow|\uparrow\rangle = 0, \langle\downarrow|\downarrow\rangle = 1.$

(c) $\langle\uparrow|\uparrow\rangle = 1, \langle\uparrow|\downarrow\rangle = -1, \langle\downarrow|\uparrow\rangle = -1, \langle\downarrow|\downarrow\rangle = 1.$

(d) $\langle\uparrow|\uparrow\rangle = 1, \langle\uparrow|\downarrow\rangle = 0, \langle\downarrow|\uparrow\rangle = 0, \langle\downarrow|\downarrow\rangle = -1.$

10. Which one of the following sets of expressions is correct?

(a) $\langle\uparrow|\hat{S}_z|\uparrow\rangle = \hbar/2,\ \langle\downarrow|\hat{S}_z|\uparrow\rangle = 0,\ \langle\uparrow|\hat{S}_z|\downarrow\rangle = 0,\ \langle\downarrow|\hat{S}_z|\downarrow\rangle = \hbar/2.$

(b) $\langle\uparrow|\hat{S}_z|\uparrow\rangle = \hbar/2,\ \langle\downarrow|\hat{S}_z|\uparrow\rangle = 0,\ \langle\uparrow|\hat{S}_z|\downarrow\rangle = 0,\ \langle\downarrow|\hat{S}_z|\downarrow\rangle = -\hbar/2.$

(c) $\langle\uparrow|\hat{S}_z|\uparrow\rangle = \hbar/2,\ \langle\downarrow|\hat{S}_z|\uparrow\rangle = -\hbar/2,\ \langle\uparrow|\hat{S}_z|\downarrow\rangle = -\hbar/2,\ \langle\downarrow|\hat{S}_z|\downarrow\rangle = \hbar/2.$

(d) $\langle\uparrow|\hat{S}_z|\uparrow\rangle = \hbar/2,\ \langle\downarrow|\hat{S}_z|\uparrow\rangle = -\hbar/2,\ \langle\uparrow|\hat{S}_z|\downarrow\rangle = -\hbar/2,\ \langle\downarrow|\hat{S}_z|\downarrow\rangle = 0.$

11. Which one of the following is the matrix representation of $\hat{S}_z$ if we choose the basis vectors in the order $|\uparrow\rangle, |\downarrow\rangle$ to construct the matrix. Each of the four elements of the matrix is a "number" and is called a matrix element.

(a) $\begin{pmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & \langle\uparrow|\hat{S}_z|\downarrow\rangle \\ \langle\downarrow|\hat{S}_z|\uparrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{pmatrix}$

(b) $\begin{pmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & -\langle\uparrow|\hat{S}_z|\downarrow\rangle \\ -\langle\downarrow|\hat{S}_z|\uparrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{pmatrix}$

(c) $\begin{pmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & \langle\uparrow|\hat{S}_z|\uparrow\rangle \\ \langle\downarrow|\hat{S}_z|\downarrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{pmatrix}$

(d) $\begin{pmatrix} \langle\uparrow|\hat{S}_z|\uparrow\rangle & \langle\downarrow|\hat{S}_z|\uparrow\rangle \\ \langle\uparrow|\hat{S}_z|\downarrow\rangle & \langle\downarrow|\hat{S}_z|\downarrow\rangle \end{pmatrix}$

12. Consider the following conversation between Kathy and Tim:

• Kathy: In order to have a diagonal matrix, the off-diagonal matrix elements, e.g., $\langle\downarrow|\hat{S}_z|\uparrow\rangle$ must be zero. Since $|\uparrow\rangle$ is an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$, $\hbar/2$ gets pulled out of the scalar product and the orthogonality of basis vectors implies $\langle\downarrow|\uparrow\rangle = 0$.

• Tim: I agree. This means that whenever we write any operator in a matrix form after choosing an orthonormal basis, the matrix will be diagonal if the basis vectors are eigenstates of that operator. This is because the operator acting on any basis vector will yield the corresponding eigenvalue which is a number and gets pulled out. The orthogonality of basis vectors then makes the off-diagonal matrix elements zero.

Do you agree with Kathy and Tim? Explain.

(a) Yes

(b) No

13. Use the matrix elements of $\hat{S}_z$ in the previous question to construct its matrix representation in the $|\uparrow\rangle$ and $|\downarrow\rangle$ basis. Which one of the following is the matrix representation of $\hat{S}_z$ if we choose the basis vectors in the order $|\uparrow\rangle, |\downarrow\rangle$ to construct the matrix.

(a) $\hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $\hbar/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $\hbar/2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(d) $\hbar/2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

14. Which one of the following is the matrix representation of $\hat{S}^2$ if we choose the basis vectors in the order $|\uparrow\rangle, |\downarrow\rangle$ to construct the matrix.

(a) $(3\hbar^2/4)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $(3\hbar^2/4)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $(3\hbar^2/4)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $(3\hbar^2/4)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

15. Choose all of the following statements that are true about the Hamiltonian $\hat{H}_0 = c\hat{S}_z$ (where c is a constant) for a spin-1/2 system:

(I) If we choose two different bases (coordinates), $\hat{H}_0$ may be a diagonal matrix in one basis but not in the other.

(II) If the basis vectors are eigenstates of $\hat{H}_0$, $\hat{H}_0$ will be a diagonal matrix.

(III) No matter what basis we choose, $\hat{H}_0$ must always be a diagonal matrix by definition.

(a) (I) only

(b) (II) only

(c) (I) and (II) only.

(d) (II) and (III) only

16. Suppose the Hamiltonian for a single spin-1/2 system is given by $\hat{H} = -(e/m)\ \hat{S}_z\ B_0$ where $\hat{S}_z$ is the z component of the spin angular momentum operator and B_0 is the strength of a uniform magnetic field applied in the $+z$ direction. Which one of the following is a basis in which $\hat{H}$ can be written as a diagonal matrix?

(a) The eigenstates of $\hat{S}^2$ but not $\hat{S}_z$

(b) The eigenstates of $\hat{S}^2$ and $\hat{S}_z$

(c) The eigenstates of the raising operator $\hat{S}_+$

(d) None of the above

17. For a spin-1/2 system with the Hamiltonian $\hat{H}_0 = C\hat{S}_z$ (where C is a constant), which one of the following is a basis that will yield $\hat{H}_0$ as a diagonal matrix?

(a) $|m_x, m_y\rangle$, simultaneous eigenstates of $\hat{S}_x$ and $\hat{S}_y$.

(b) $|s, m_z\rangle$, simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$.

(c) $|s, m_x\rangle$, simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_x$.

(d) $|s, m_n\rangle$, simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_n$ where n points in an arbitrary direction in space.

Reviewing raising and lowering operators for a single spin system

Notation: In the algebra of angular momentum, the raising and lowering operators have the following effect on the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$:

$\hat{S}_{\pm}|sm_z\rangle = \hbar\sqrt{s(s+1) - m_z(m_z \pm 1)}|s(m_z \pm 1)\rangle$

The raising operator acting on the highest m_z state annilates it (prefactor $\sqrt{s(s+1) - m_z(m_z + 1)} = 0$).

The lowering operator acting on the lowest m_z state annilates it (prefactor $\sqrt{s(s+1) - m_z(m_z - 1)} = 0$).

18. Which one of the following is the correct expression for $S_{\pm}|Sm_z\rangle$ for a spin-1/2 system?
    (a) $\hat{S}_+|\uparrow\rangle = 0, \hat{S}_+|\downarrow\rangle = \hbar|\uparrow\rangle, \hat{S}_-|\downarrow\rangle = 0, \hat{S}_-|\uparrow\rangle = \hbar|\downarrow\rangle$
    (b) $\hat{S}_+|\uparrow\rangle = 0, \hat{S}_+|\downarrow\rangle = \hbar/2|\uparrow\rangle, \hat{S}_-|\downarrow\rangle = 0, \hat{S}_-|\uparrow\rangle = \hbar/2|\downarrow\rangle$
    (c) $\hat{S}_+|\uparrow\rangle = 0, S_+|\downarrow\rangle = -\hbar|\uparrow\rangle, S_-|\downarrow\rangle = 0, S_-|\uparrow\rangle = -\hbar|\downarrow\rangle$
    (d) $\hat{S}_+|\uparrow\rangle = 0, \hat{S}_+|\downarrow\rangle = \hbar|\uparrow\rangle, \hat{S}_-|\downarrow\rangle = 0, \hat{S}_-|\uparrow\rangle = -\hbar|\downarrow\rangle$

19. Which one of the following is the correct expression for $\langle\uparrow|\hat{S}_+|\uparrow\rangle$ for a spin-1/2 system?
    (a) 0
    (b) $\hbar$
    (c) $\hbar/2$
    (d) $-\hbar$

20. Which one of the following is the correct expression for $\langle\uparrow|\hat{S}_+|\downarrow\rangle$ for a spin-1/2 system?
    (a) 0
    (b) $\hbar$
    (c) $\hbar/2$
    (d) $-\hbar$

21. Which one of the following is the correct expression for the matrix representation of the raising operator $\hat{S}_+$ of a spin-1/2 system if the basis vectors are given by the ordered list $\{|\uparrow\rangle, |\downarrow\rangle\}$ (the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$)?

(Hint: Use the matrix elements you calculated above.)

(a) $\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $\hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(d) $\hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

22. Which one of the following is the correct expression for $\langle\uparrow|\hat{S}_-|\uparrow\rangle$ for a spin-1/2 system?

(a) 0

(b) $\hbar$

(c) $\hbar/2$

(d) $-\hbar$

23. Which one of the following is the correct expression for $\langle\downarrow|\hat{S}_-|\uparrow\rangle$ for a spin-1/2 system?

(a) 0

(b) $\hbar$

(c) $\hbar/2$

(d) $-\hbar$

24. Which one of the following is the correct expression for the matrix representation of the lowering operator $\hat{S}_-$ of a spin-1/2 system if the basis vectors are given by the ordered list $\{|\uparrow\rangle, |\downarrow\rangle\}$ (the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$)?

(Hint: Use the matrix elements you calculated above.)

(a) $\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $\hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(d) $\hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Warm-up: Post-test

Notation: For a spin-one-half (spin-1/2) system, the spin quantum number $s = 1/2$ and the quantum number for the z component of the spin is $m_z = \pm 1/2$. Therefore, if we choose the orthonormal basis vectors to be the eigenstates of the operators $\hat{S}^2$ and $\hat{S}_z$, we can denote them as $|s, m_z\rangle$: $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$.

• Since the spin quantum number $s = 1/2$ is fixed, we can simplify the notation for the basis vectors as $|s, m_z\rangle \equiv |m_z\rangle \equiv |\pm 1/2\rangle$ where $m_z = \pm 1/2$.

• Another common notation for the basis vectors $|s, m_z\rangle$ for a spin-1/2 system is $|1/2\rangle \equiv |\uparrow\rangle$ and $|-1/2\rangle \equiv |\downarrow\rangle$.

• The eigenvalue equations for the operators $\hat{S}_z$ and $\hat{S}^2$ are given by:

$\hat{S}_z|\uparrow\rangle = (\hbar/2)|\uparrow\rangle$, and $\hat{S}_z|\downarrow\rangle = (-\hbar/2)|\downarrow\rangle$

$\hat{S}^2|\uparrow\rangle = \hbar^2 s(s+1)|\uparrow\rangle = (\hbar^2/2)(1/2+1)|\uparrow\rangle = (3\hbar^2/4)|\uparrow\rangle$,

and $\hat{S}^2|\downarrow\rangle = \hbar^2 s(s+1)|\downarrow\rangle = (\hbar^2/2)(1/2+1)|\downarrow\rangle = (3\hbar^2/4)|\downarrow\rangle$

In the algebra of angular momentum, the raising and lowering operators have the following effect on the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$:

$\hat{S}_\pm|sm_z\rangle = \hbar\sqrt{s(s+1) - m_z(m_z \pm 1)}|s(m_z \pm 1)\rangle$

The raising operator acting on the highest m_z state annilates it (prefactor $\sqrt{s(s+1) - m_z(m_z+1)} = 0$).

The lowering operator acting on the lowest m_z state annilates it (prefactor $\sqrt{s(s+1) - m_z(m_z-1)} = 0$).

The matrix representations for $|\uparrow\rangle$ and $|\downarrow\rangle$ are given by $|\uparrow\rangle \doteq \binom{1}{0}$, and $|\downarrow\rangle \doteq \binom{0}{1}$.

1. Write down the Hamiltonian $\hat{H}_0 = constant(\hat{S}^2 - \hat{S}_z^2)$ for a spin-1/2 system in a matrix form if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix.

2. Write down $\hat{S}^2$ for a spin-1/2 system in a matrix form if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix.

3. Write down the expression for $\langle\downarrow|\hat{S}_-|\uparrow\rangle$ for a spin-1/2 system.

4. Write down the expression for the matrix representation of the lowering operator $\hat{S}_-$ of a spin-1/2 system if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix.

Pretest

Two spin-1/2 systems (with the spin quantum numbers $s_1 = 1/2$ and $s_2 = 1/2$) at fixed locations in space (only consider spin degrees of freedom) interact with each other, and with a uniform magnetic field $\vec{B}$ pointing in the $+z$ direction. When the magnetic field is off, the interaction between the spins is given by the Hamiltonian:

$$\hat{H}_1 = (4E_0/\hbar^2)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = (2E_0/\hbar^2)(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2) \tag{1}$$

where $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ and E_0 is a constant.
The magnetic field interacts with each spin as follows:

$$\hat{H}_2 = -\mu(\hat{\vec{S}}_1 \cdot \vec{B} + \hat{\vec{S}}_2 \cdot \vec{B}) \tag{2}$$

(a) Write down a complete set of basis vectors for the vector space of a system of two spin-1/2 particles. Explain the labels you are using to identify your basis states.
(b) Express the Hamiltonian $\hat{H}_1$ in the basis you have chosen. (Write it down as an $N \times N$ matrix.)
(c) Express the Hamiltonian $\hat{H}_2$ in the basis you have chosen.
(d) Are both $\hat{H}_1$ and $\hat{H}_2$ diagonal matrices in the basis you chose?

Posttest

A spin-1/2 system and a spin-one system (with the spin quantum numbers $s_1 = 1/2$ and $s_2 = 1$) at fixed locations in space (only consider spin degrees of freedom) interact with each other, and with a uniform magnetic field $\vec{B}$ pointing in the $+z$ direction. When the magnetic field is off, the interaction between the spins is given by the Hamiltonian:

$$\hat{H}_1 = (4E_0/\hbar^2)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = (2E_0/\hbar^2)(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2) \tag{3}$$

where $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ and E_0 is a constant.
The magnetic field interacts with each spin as follows:

$$\hat{H}_2 = -\mu(\hat{\vec{S}}_1 \cdot \vec{B} + \hat{\vec{S}}_2 \cdot \vec{B}) \tag{4}$$

(a) Write down a complete set of basis vectors for the vector space of a system of a spin-1/2 particle and a spin-one particle. Explain the labels you are using to identify your basis states.
(b) Express the Hamiltonian $\hat{H}_1$ in the basis you have chosen. (Write it down as an $N \times N$ matrix.)
(c) Express the Hamiltonian $\hat{H}_2$ in the basis you have chosen.
(d) Are both $\hat{H}_1$ and $\hat{H}_2$ diagonal matrices in the basis you chose?

The goal of this tutorial on addition of angular momentum preliminaries is to review the basics of

- *product space for systems with various angular momenta (spin or orbital) specified via their angular momentum quantum numbers*
- *vector space (product space) of two spin-one-half (spin-1/2) systems in the uncoupled and uncoupled representations*
- *generalizing to compute the dimensionality, basis vectors and operator matrices for other product spaces in the uncoupled and coupled representations*

At the end of this tutorial, you should be able to do problems of the type given in the pretest. In particular, you should be able to find the dimensionality of the product space of two angular momenta (product space of two spin angular momenta or product space of two orbital angular momenta or product space of an orbital angular momentum and a spin angular momentum given the corresponding angular momentum quantum numbers), find a complete set of basis vectors for the product space in the uncoupled representation, find all matrix elements of any operator in the product space in the chosen basis and construct the operator matrix. It is important that you work through this tutorial with a blank sheet of paper and pen so that you can write down important ideas you learned as you go along. It is recommended that you work through the tutorial on a single spin system before working on this tutorial.

Product Space for two spin-1/2 systems

1. *Consider the following conversation between Andy and Caroline:*
   - *Andy: In the pretest, we have to write the basis vectors for the vector space of two spin-1/2 systems. Isn't the vector space of two spin-1/2 systems two dimensional since each is a spin-1/2 system?*
   - *Caroline: No. The vector space of two spin-1/2 systems is a four dimensional product space.*

   With whom do you agree?
   (a) *Andy*
   (b) *Caroline*
   (c) *Both*

2. *What is the dimensionality of the product space of the spin degrees of freedom for two spin-1/2 systems?*
   (a) *2*
   (b) *2+2=4*
   (c) $2 \times 2 = 4$
   (d) *2, 3 or 4*

3. *How many basis vectors do you need in this product space to represent a general state (vector)?*
   (a) *2*
   (b) *4*
   (c) *2, 3 or 4*
   (d) *infinite*

4. *Which one of the following is an appropriate notation for the basis vectors in the product space of two spin-1/2 systems?*
   (a) $|s_1, m_{1z}\rangle$
   (b) $|s_2, m_{2z}\rangle$
   (c) $|s_1, m_{1z}\rangle \otimes |s_2, m_{2z}\rangle$
   (d) *none of the above*

   Hint: The symbol $\otimes$ *is commonly used for the direct product of states for each of the two systems.*

5. *Is the phrase "addition of angular momentum" confusing since it involves finding the product space for the vector spaces of the individual angular momenta where the dimensionality of the product space is the product (NOT the sum) of the dimensions of individual vector spaces? If you are working on this tutorial in a group, discuss this issue with others in your group.*

Product Space for two spin-1/2 (spin-1/2) systems: Uncoupled representation

Notation and conventions for two spin-1/2 systems in uncoupled representation:

- *Since the spin quantum numbers $S_1 = 1/2$ and $S_2 = 1/2$ are fixed, we can represent orthonormal basis vectors in a product state (by explicitly stating only m_{1z} and m_{2z}, the quantum numbers associated with the z component of S_1 and S_2 respectively) as $|s_1, m_{1z}\rangle \otimes |s_2, m_{2z}\rangle \equiv |m_{1z}\rangle \otimes |m_{2z}\rangle$ or $|m_{1z}m_{2z}\rangle$ for short where label 1 is for the first system and label 2 for the second system.*
- *Since $m_{1z} = \pm 1/2$ and $m_{2z} = \pm 1/2$ there are four basis vectors in the product space of two spin-1/2 systems: $|1/2, 1/2\rangle \otimes |1/2, 1/2\rangle \equiv |\uparrow\uparrow\rangle \equiv |\uparrow\rangle_1 |\uparrow\rangle_2$, $|1/2, 1/2\rangle \otimes |1/2, -1/2\rangle \equiv |\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$, $|1/2, -1/2\rangle \otimes |1/2, 1/2\rangle \equiv |\downarrow\uparrow\rangle \equiv |\downarrow\rangle_1 |\uparrow\rangle_2$, and $|1/2, -1/2\rangle \otimes |1/2, -1/2\rangle \equiv |\downarrow\downarrow\rangle \equiv |\downarrow\rangle_1 |\downarrow\rangle_2$.*
- *Note that although we drop the labels 1 and 2 for the two spin-1/2 systems in this more compact notation, e.g., $|\downarrow\uparrow\rangle \equiv |\downarrow\rangle_1 |\uparrow\rangle_2$, the spin operators corresponding to system 1 will act on the state represented by the first symbol (in this case $|\downarrow\rangle$ which is an eigenstate of S_{1z} with eigenvalue $-\hbar/2$) and the spin operators for system 2 will act on the state represented by the second symbol $|\uparrow\rangle$.*
- *The following examples further illustrate why "uncoupled representation" is a suitable name for such a basis in the product space:*

$\hat{S}_{1z}|\uparrow\downarrow\rangle = \hat{S}_{1z}|\uparrow\rangle_1|\downarrow\rangle_2 = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 = ((\hbar/2)|\uparrow\rangle_1)|\downarrow\rangle_2 = (\hbar/2)|\uparrow\rangle_1|\downarrow\rangle_2 = (\hbar/2)|\uparrow\downarrow\rangle$

$\hat{S}_{1z}\hat{S}_{2z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)(\hat{S}_{2z}|\downarrow\rangle_2) = ((\hbar/2)|\uparrow\rangle_1)((-\hbar/2)|\downarrow\rangle_2) = -(\hbar/2)^2|\uparrow\downarrow\rangle$

The matrix representations for $|\uparrow\rangle$ and $|\downarrow\rangle$ are given by $|\uparrow\rangle \doteq \binom{1}{0}$, and $|\downarrow\rangle \doteq \binom{0}{1}$.

6. *Consider the following statements about the "uncoupled" representation for a two-spin system:*

   (I) We can label the basis vectors using quantum numbers m_{1z} and m_{2z} for the two spins.

   (II) The operators for one spin do not act on states (vectors) for the other spin.

   (III) It is possible to put labels 1 and 2 on the states and the operators to differentiate which operators act on which states.

   Choose all of the above statements that are true.

   (a) *(I) and (II) only*
   (b) *(I) and (III) only*
   (c) *(II) and (III) only*
   (d) *(I), (II) and (III).*

7. *Consider the following statements based upon what you have learned so far in this tutorial:*

   (I) $|\uparrow\uparrow\rangle$ *is an eigenstate of* $\hat{S}_{1z}$ *and* $\hat{S}_{2z}$ *but not* $\hat{S}_1^2$ *or* $\hat{S}_2^2$.

   (II) $|\uparrow\uparrow\rangle$ *is an eigenstate of* $\hat{S}_{1z}$, $\hat{S}_{2z}$, $\hat{S}_1^2$ *and* $\hat{S}_2^2$.

   (III) $|\uparrow\uparrow\rangle$ *is an eigenstate of* $\hat{S}_{1z}$, $\hat{S}_{2z}$, $\hat{\vec{S}}_1$ *and* $\hat{\vec{S}}_2$.

   Choose all of the above statements that are true.

   (a) *(I) only*

   (b) *(II) only*

   (c) *(III) only*

   (d) *(I) and (III) only*

8. *Consider the following statements based upon what you have learned so far in this tutorial:*

   (I) $|1/2, 1/2\rangle \otimes |1/2, -1/2\rangle \equiv |\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$

   (II) $|1/2, -1/2\rangle \otimes |1/2, -1/2\rangle \equiv |\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$

   (III) $|1/2, 1/2\rangle \otimes |-1/2, -1/2\rangle \equiv |\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$

   Choose all of the above statements that are true.

   (a) *(I) only*

   (b) *(II) only*

   (c) *(III) only*

   (d) *None of the above*

9. *Consider the following statements. For two spin systems, the simultaneous eigenstates of* $\hat{S}_{1z}$, $\hat{S}_{2z}$, $\hat{S}_1^2$, *and* $\hat{S}_2^2$ *are called the "uncoupled" representation because*

   (I) the basis vectors of the type $|\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$ *can be decoupled into products of states involving systems 1 and 2 separately.*

   (II) the operators related to each spin system act on their own states, e.g., $\hat{S}_{1z}$ *acts on* $|\uparrow\rangle_1$ *in the product state* $|\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$.

   (III) someone thought of an inappropriate name for this representation.

   Choose all of the above statements that are true.

   (a) *(I) only*

   (b) *(II) only*

   (c) *(III) only*

   (d) *(I) and (II) only*

10. *Which one of the following statements is correct?*

    (a) $|\uparrow\downarrow\rangle$ *and* $|\downarrow\uparrow\rangle$ *are essentially the same states (vectors).*

    (b) $|\uparrow\downarrow\rangle$ *and* $|\downarrow\uparrow\rangle$ *are distinctly different states (vectors).*

    (c) $|\uparrow\downarrow\rangle = -|\downarrow\uparrow\rangle$

    (d) $|\uparrow\downarrow\rangle$ *and* $|\downarrow\uparrow\rangle$ *may be the same or different states depending on the choice of the operator (e.g.,* $\hat{S}_z$ *vs.* $\hat{S}_x$*).*

11. *Which one of the following is a correct matrix representation of the basis vectors* $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, *and* $|\downarrow\downarrow\rangle$ *for two spin-1/2 systems?*

(a) $\binom{1}{1}$, $\binom{1}{0}$, $\binom{0}{1}$, *and* $\binom{0}{0}$

(b) $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, *and* $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$,

(c) $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, *and* $\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$

(d) $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$, *and* $\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$.

12. *Which one of the following is correct?*

(a) $\hat{S}_{1z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 = (\hbar/2)|\uparrow\downarrow\rangle$

(b) $\hat{S}_{1z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 = (-\hbar/2)|\uparrow\downarrow\rangle$

(c) $\hat{S}_{1z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 = \hbar|\uparrow\downarrow\rangle$

(d) $\hat{S}_{1z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 = 0|\uparrow\downarrow\rangle$

13. *Which one of the following is correct?*

(a) $\hat{S}_{2z}|\uparrow\downarrow\rangle = |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = (\hbar/2)|\uparrow\downarrow\rangle$

(b) $\hat{S}_{2z}|\uparrow\downarrow\rangle = |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = (-\hbar/2)|\uparrow\downarrow\rangle$

(c) $\hat{S}_{2z}|\uparrow\downarrow\rangle = |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = \hbar|\uparrow\downarrow\rangle$

(d) $\hat{S}_{2z}|\uparrow\downarrow\rangle = |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = 0|\uparrow\downarrow\rangle$

14. *Which one of the following is correct?*

(a) $(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = \hat{S}_{1z}|\uparrow\downarrow\rangle + \hat{S}_{2z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 + |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = 0$

(b) $(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = \hat{S}_{1z}|\uparrow\downarrow\rangle + \hat{S}_{2z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 + |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = (\hbar/2)|\uparrow\downarrow\rangle$

(c) $(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = \hat{S}_{1z}|\uparrow\downarrow\rangle + \hat{S}_{2z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 + |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = (\hbar/2)^2|\uparrow\downarrow\rangle$

(d) $(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = \hat{S}_{1z}|\uparrow\downarrow\rangle + \hat{S}_{2z}|\uparrow\downarrow\rangle = (\hat{S}_{1z}|\uparrow\rangle_1)|\downarrow\rangle_2 + |\uparrow\rangle_1(\hat{S}_{2z}|\downarrow\rangle_2) = \hbar|\uparrow\downarrow\rangle$

15. *Which one of the following is correct? (Hint: Use the results of the previous question and then take the scalar product.)*

(a) $\langle\uparrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = 0$

(b) $\langle\uparrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = \hbar/2$

(c) $\langle\uparrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = \hbar$

(d) $\langle\uparrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\downarrow\rangle = (\hbar/2)^2$

16. *Which one of the following is correct?*

(a) $\langle\uparrow\uparrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\uparrow\rangle = 0$

(b) $\langle\uparrow\uparrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\uparrow\rangle = \hbar/2$

(c) $\langle\uparrow\uparrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\uparrow\rangle = \hbar$

(d) $\langle\uparrow\uparrow|(\hat{S}_{1z}+\hat{S}_{2z})|\uparrow\uparrow\rangle = (\hbar/2)^2$

17. *Which one of the following is correct?*

(a) $\langle\downarrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\downarrow\downarrow\rangle = 0$

(b) $\langle\downarrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\downarrow\downarrow\rangle = -\hbar/2$

(c) $\langle\downarrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\downarrow\downarrow\rangle = -\hbar$

(d) $\langle\downarrow\downarrow|(\hat{S}_{1z}+\hat{S}_{2z})|\downarrow\downarrow\rangle = (\hbar/2)^2$

18. *Which one of the following is correct? (Note: The operator involves $\hat{S}_{1z}^2$ not $\hat{S}_1^2$)*

(a) $\langle\uparrow\downarrow|(\hat{S}_{1z}^2+\hat{S}_{2z}^2)|\uparrow\downarrow\rangle = 0$

(b) $\langle\uparrow\downarrow|(\hat{S}_{1z}^2+\hat{S}_{2z}^2)|\uparrow\downarrow\rangle = \hbar^2$

(c) $\langle\uparrow\downarrow|(\hat{S}_{1z}^2+\hat{S}_{2z}^2)|\uparrow\downarrow\rangle = 2\hbar^2$

(d) $\langle\uparrow\downarrow|(\hat{S}_{1z}^2+\hat{S}_{2z}^2)|\uparrow\downarrow\rangle = 2(\hbar/2)^2$

Constructing $\hat{H}_2$ matrix for two spin-1/2 systems in the "uncoupled" representation

19. *In the "uncoupled" basis $|m_{1z}m_{2z}\rangle$ in the product space, which one of the following matrices represents the Hamiltonian $\hat{H}_2 = \mu(\hat{\vec{S}}_1\cdot\vec{B}+\hat{\vec{S}}_2\cdot\vec{B})$ where the uniform magnetic field is in the z direction. Choose the basis vectors in the order $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ to write $\hat{H}_2$ in the matrix form. Hint: Use the matrix elements you calculated in previous questions.*

(a) $\mu B\begin{pmatrix}\hbar/2 & 0\\ 0 & -\hbar/2\end{pmatrix}$

(b) $\mu B\begin{pmatrix}\hbar & 0\\ 0 & -\hbar\end{pmatrix}$

(c) $\mu B\begin{pmatrix}\hbar/2 & 0 & 0 & 0\\ 0 & 0 & -\hbar/2 & 0\\ 0 & \hbar & 0 & 0\\ 0 & 0 & 0 & \hbar\end{pmatrix}$

(d) $\mu B\begin{pmatrix}\hbar & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\hbar\end{pmatrix}$

20. *In order to construct the matrix $\hat{H}_2$ above, if we change the order in which we write the basis vectors to $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$, which one of the following is the new matrix representation of $\hat{H}_2$?*

(a) $\mu B \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hbar & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar \end{pmatrix}$

(b) $\mu B \begin{pmatrix} \hbar & 0 & 0 & 0 \\ 0 & -\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) It will remain the same as in the previous question.

(d) There will be some non-zero off-diagonal matrix elements in the new matrix.

21. *Choose all of the following statements that are true.*

$\hat{H}_2$ above can be written as a diagonal matrix in the uncoupled representation because

(I) the basis vectors are eigenstates of $\hat{H}_2$ because $\hat{H}_2$ commutes with the operators $\hat{S}_{1z}$, $\hat{S}_{2z}$, $\hat{S}_1^2$ and $\hat{S}_2^2$.

(II) $\hat{H}_2$ is a Hamiltonian operator which must be diagonal no matter what basis you choose.

(III) we are dealing with spin-1/2 systems. $\hat{H}_2$ will not be diagonal if we had two spin-one systems.

(a) *(I) only*

(b) *(II) only*

(c) *(III) only*

(d) *(I) and (II) only*

Constructing $\hat{H}_1$ for two spin-1/2 systems in the "uncoupled" representation

The Hamiltonian $\hat{H}_1$ for a system of two spin-1/2 particles is $\hat{H}_1 = (4E_0/\hbar^2)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = (2E_0/\hbar^2)(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)$. We want to write $\hat{H}_1$ in a matrix form in the "uncoupled" representation in which the basis vectors are $|m_{1z}, m_{2z}\rangle$, the simultaneous eigenstates of $\hat{S}_1^2$, $\hat{S}_{1z}$, $\hat{S}_2^2$, and $\hat{S}_{2z}$.

22. *Consider the following statements from Pria and Mira:*

- *Pria: Is $(4E_0/\hbar^2)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ or $(2E_0/\hbar^2)(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)$ the more convenient form for writing $\hat{H}_1$ in matrix form in the uncoupled representation without using a table?*
- *Mira: Since the basis vectors $|m_{1z}, m_{2z}\rangle$ are not the eigenstates of $\hat{H}_1$, we have to be careful. It is the form $\hat{H}_1 = (4E_0/\hbar^2)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ that is more useful because we can write $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$. Then we can write the x and y components of spin in terms of the raising and lowering operators and we know how they act on $|m_{1z}, m_{2z}\rangle$.*

Do you agree with Mira?

(a) Yes, (b) No

23. *Define the raising and lowering operators for a single spin-1/2 system as* $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$ *and* $\hat{S}_- = \hat{S}_x - i\hat{S}_y$. *Which one of the following gives the correct values for* $\hat{S}_x$ *and* $\hat{S}_y$*?*

(a) $\hat{S}_x = (\hat{S}_+ + \hat{S}_-)/2,\ \hat{S}_y = (\hat{S}_+ - \hat{S}_-)/2i$

(b) $\hat{S}_x = (\hat{S}_+ - \hat{S}_-)/2,\ \hat{S}_y = (\hat{S}_+ - \hat{S}_-)/2i$

(c) $\hat{S}_x = (\hat{S}_+ + \hat{S}_-)/2,\ \hat{S}_y = (\hat{S}_+ + \hat{S}_-)/2i$

(d) $\hat{S}_x = (\hat{S}_+ - \hat{S}_-)/2,\ \hat{S}_y = (\hat{S}_+ + \hat{S}_-)/2i$

24. *Define the raising and lowering operators for each spin, e.g., for the first spin as* $\hat{S}_{1+} = \hat{S}_{1x} + i\hat{S}_{1y}$ *and* $\hat{S}_{1-} = \hat{S}_{1x} - i\hat{S}_{1y}$.

Choose all of the following expressions that are correct for $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$. *These expressions will be helpful in writing* $\hat{H}_1$ *above in the "uncoupled" basis:*

(I) $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$

(II) $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = (\hat{S}_{1-}\hat{S}_{2+} + \hat{S}_{1+}\hat{S}_{2-})/2 + \hat{S}_{1z}\hat{S}_{2z}$

(III) $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = (\hat{S}_{1-}\hat{S}_{2+} + \hat{S}_{1+}\hat{S}_{2-})/2 - \hat{S}_{1z}\hat{S}_{2z}$

(a) (I) only

(b) (II) only

(c) (III) only

(d) (I) and (II) only

(e) (I) and (III) only

Summary: As can be seen above, in order to compute $\hat{H}_1$ *(which involves* $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$*) in the basis of the simultaneous eigenstates of* $\hat{S}_{1z}$ *and* $\hat{S}_{2z}$*, it is useful to write* $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ *in terms of the raising and lowering operators.*

In this section, we will review the properties of the raising and lowering operators for a single spin-1/2 system before dealing with two spin-1/2 systems.

25. *Consider two spin-1/2 systems. Which one of the following is correct?*

(a) $\hat{S}_{1-}\hat{S}_{2+}|\uparrow\uparrow\rangle = (\hat{S}_{1-}|\uparrow\rangle_1)(\hat{S}_{2+}|\uparrow\rangle_2) = 0$

(b) $\hat{S}_{1-}\hat{S}_{2+}|\uparrow\uparrow\rangle = (\hat{S}_{1-}|\uparrow\rangle_1)(\hat{S}_{2+}|\uparrow\rangle_2) = \hbar^2|\uparrow\uparrow\rangle$

(c) $\hat{S}_{1-}\hat{S}_{2+}|\uparrow\uparrow\rangle = (\hat{S}_{1-}|\uparrow\rangle_1)(\hat{S}_{2+}|\uparrow\rangle_2) = 2\hbar^2|\uparrow\uparrow\rangle$

(d) None of the above.

26. *Consider two spin-1/2 systems. Which one of the following is correct?*

(a) $\hat{S}_{1-}\hat{S}_{2+}|\downarrow\downarrow\rangle = (\hat{S}_{1-}|\downarrow\rangle_1)(\hat{S}_{2+}|\downarrow\rangle_2) = 0$

(b) $\hat{S}_{1-}\hat{S}_{2+}|\downarrow\downarrow\rangle = (\hat{S}_{1-}|\downarrow\rangle_1)(\hat{S}_{2+}|\downarrow\rangle_2) = \hbar^2|\downarrow\downarrow\rangle$

(c) $\hat{S}_{1-}\hat{S}_{2+}|\downarrow\downarrow\rangle = (\hat{S}_{1-}|\downarrow\rangle_1)(\hat{S}_{2+}|\downarrow\rangle_2) = 2\hbar^2|\downarrow\downarrow\rangle$

(d) None of the above.

27. *Consider two spin-1/2 systems. Which one of the following is correct?*
(a) $\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = (\hat{S}_{1-}|\uparrow\rangle_1)(\hat{S}_{2+}|\downarrow\rangle_2) = 0$
(b) $\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = (\hat{S}_{1-}|\uparrow\rangle_1)(\hat{S}_{2+}|\downarrow\rangle_2) = \hbar^2|\downarrow\uparrow\rangle$
(c) $\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = (\hat{S}_{1-}|\uparrow\rangle_1)(\hat{S}_{2+}|\downarrow\rangle_2) = 2\hbar^2|\downarrow\uparrow\rangle$
(d) None of the above.

28. *Consider two spin-1/2 systems. Which one of the following scalar products is correct?*
(a) $\langle\downarrow\uparrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = (\langle\downarrow|\hat{S}_{1-}|\uparrow\rangle_1)(\langle\uparrow|\hat{S}_{2+}|\downarrow\rangle_2) = 0$
(b) $\langle\downarrow\uparrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = (\langle\downarrow|\hat{S}_{1-}|\uparrow\rangle_1)(\langle\uparrow|\hat{S}_{2+}|\downarrow\rangle_2) = \hbar^2$
(c) $\langle\downarrow\uparrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = (\langle\downarrow|\hat{S}_{1-}|\uparrow\rangle_1)(\langle\uparrow|\hat{S}_{2+}|\downarrow\rangle_2) = 2\hbar^2$
(d) None of the above.

29. *Consider two spin-1/2 systems. Which one of the following scalar products is correct? (Hint: Note that the "bra" states are different here than in the previous question.)*
(a) $\langle\uparrow\downarrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = 0$
(b) $\langle\uparrow\downarrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = \hbar^2$
(c) $\langle\uparrow\downarrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle = 2\hbar^2$
(d) None of the above.

30. *Consider the following statements from Pria and Mira:*
- *Pria: The Hamiltonian $\hat{H}_1$ will not be a diagonal matrix if we choose the basis vectors to be $|m_{1z}, m_{2z}\rangle$, the simultaneous eigenstates of $\hat{S}_1^2$, $\hat{S}_{1z}$, $\hat{S}_2^2$, and $\hat{S}_{2z}$.*
- *Mira: I agree. $\hat{H}_1$ has terms involving the x and y components of spin because $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$. Therefore, $\hat{H}_1|m_{1z}, m_{2z}\rangle$ will not simply give a "number" multiplied by the same state $|m_{1z}, m_{2z}\rangle$ back.*
- *Pria: And if you replace the terms involving the x and y components of spin with the raising and lowering operators, the states $|m_{1z}, m_{2z}\rangle$ will change when the raising and lowering operators act on them. Then, when you take a scalar product with $\langle m'_{1z}, m'_{2z}|$ to find a matrix element of $\hat{H}_1$ you will obtain non-zero off-diagonal terms.*

Do you agree with Pria and Mira? (It may be helpful to work out at least one off-diagonal matrix element of the type $\langle m'_{1z}, m'_{2z}|\hat{S}_{1-}\hat{S}_{2+}|m_{1z}, m_{2z}\rangle$, e.g., $\langle\downarrow\uparrow|\hat{S}_{1-}\hat{S}_{2+}|\uparrow\downarrow\rangle$. Make sure you understand why this term is an off-diagonal term in $\hat{H}_1$ by observing whether the "bra" and "ket" vectors are the same or different).
(a) Yes.

(b) No.

31. Choose all of the following statements that are correct:

(I) $\hat{H}_1$ is an off-diagonal matrix if the basis vectors are the simultaneous eigenstates of $\hat{S}_1^2$, $\hat{S}_{1z}$, $\hat{S}_2^2$, and $\hat{S}_{2z}$.

(II) It is possible to put $\hat{H}_1$ into a diagonal matrix form in a suitable basis but the basis vectors will <u>not</u> be the eigenstates of $\hat{S}_1^2$, $\hat{S}_{1z}$, $\hat{S}_2^2$, and $\hat{S}_{2z}$.

(III) The basis vectors can be chosen to be the simultaneous eigenstates of $\hat{S}_{1x}$ and $\hat{S}_{1z}$.

(a) (I) only

(b) (II) only

(c) (I) and (II) only

(d) (I), (II) and (III).

32. Choose the basis vectors in the order $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ to write $\hat{H}_1$ in the matrix form. Which one of the following is the correct matrix representation of the Hamiltonian $\hat{H}_1 = (4E_0/\hbar^2)\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ in the product space? (Hint: Write $\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ in terms of the raising and lowering operators of the two spins before invoking the matrix elements you calculated in the previous questions.)

(a) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & -E_0 & 2E_0 & 0 \\ 0 & 2E_0 & -E_0 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix}$

(b) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & 0 & 2E_0 & 0 \\ 0 & 2E_0 & 0 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix}$

(c) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & 0 & E_0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix}$

(d) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix}$

33. Consider the following statements. $\hat{H}_1$ <u>cannot</u> be written as a diagonal matrix in the uncoupled representation because

(I) $\hat{H}_1$ does not commute with the operators $\hat{S}_{1z}$ and $\hat{S}_{2z}$ whose eigenstates are the basis vectors in the uncoupled representation.

(II) $\hat{H}_1$ is a Hamiltonian operator which can never be diagonal no matter what basis you choose.

(III) we are dealing with two spin-1/2 systems. $\hat{H}_1$ will be diagonal if we had two spin-one systems.

Choose all of the above statements that are true.

(a) (I) only

(b) (II) only

(c) (III) only

(d) *(I) and (III) only*

Summary of product space of two spin-1/2 systems in the "uncoupled" representation: *We learned that the vector space for two spin-1/2 systems ($S_1 = 1/2$ and $S_2 = 1/2$) is a four dimensional (product space). So far we chose the "uncoupled" representation in which the basis vectors $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are simultaneous eigenstates of $\hat{S}_1^2$, $\hat{S}_2^2$ and $\hat{S}_{1z}$ and $\hat{S}_{2z}$. We learned how to write the two parts of the Hamiltonian $\hat{H}_1$ and $\hat{H}_2$ in this basis.*

34. Consider the following conversation between Andy and Caroline:

- *Andy: For the question in the pretest about choosing a basis for two spin-1/2 systems, we do not necessarily have to choose a basis in the product space which consists of eigenstates of $\hat{S}_{1z}$ and $\hat{S}_{2z}$.*
- *Caroline: I disagree. We must choose a basis in the product space such that the basis vectors are eigenstates of $\hat{S}_{1z}$ and $\hat{S}_{2z}$.*

With whom do you agree? Explain.

(a) Andy

(b) Caroline

(c) Both

Note 1: We can choose our basis vectors (coordinates) according to our convenience. The basis vectors we have chosen so far are simultaneous eigenstates of $\hat{S}_1^2$, $\hat{S}_2^2$ and $\hat{S}_{1z}$ and $\hat{S}_{2z}$. This representation is often called the "uncoupled" representation because the operators for system 1 and system 2 are kept separate.

Note 2: Later, we will choose basis vectors in the "coupled" representation, for which they are simultaneous eigenstates of $\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2$, $\hat{S}_1^2$, $\hat{S}_2^2$ and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$. If the operators corresponding to the physical observable (e.g., Hamiltonian operator corresponding to the energy) can be put into a diagonal matrix form in a particular basis, that basis may be more convenient than others for that problem.

Generalizing to product space of three spin-1/2 systems in the "uncoupled" representation

35. *What is the dimensionality of the spin-space of three spin-1/2 systems?*
    (a) 2
    (b) 2+2+2=6
    (c) $3^2 = 9$
    (d) $2^3 = 8$

36. *How many basis vectors are needed for the spin-space of three spin-1/2 systems?*
    (a) 8
    (b) 6
    (c) 4
    (d) 2

37. *Which one of the following is a complete set of basis vectors (in the "uncoupled representation") for the spin degrees of freedom of three spin-1/2 systems?*
    (a) $|\uparrow\rangle$, $|\downarrow\rangle$
    (b) $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$
    (c) $|\uparrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$, $|\downarrow\uparrow\downarrow\rangle$, $|\uparrow\downarrow\downarrow\rangle$, $|\downarrow\downarrow\uparrow\rangle$, $|\downarrow\downarrow\downarrow\rangle$
    (d) $|\uparrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$, $|\uparrow\downarrow\downarrow\rangle$, $|\downarrow\uparrow\uparrow\rangle$, $|\downarrow\downarrow\uparrow\rangle$, $|\uparrow\downarrow\uparrow\rangle$, $|\downarrow\downarrow\downarrow\rangle$, $|\downarrow\uparrow\downarrow\rangle$

38. *Which one of the following is true about the matrix representation of the operator $\hat{S}_z = \hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}$ for three spin-1/2 systems in the uncoupled representation you chose in the previous question.*
    (a) It will be an 8×8 diagonal matrix (i.e., only the diagonal matrix elements are nonzero).
    (b) It will be an 8×8 non-diagonal matrix.
    (c) It will be a 8 column vector.
    (d) It will be a 8 row vector.

39. *Which one of the following is true?*
    (a) $\langle\uparrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = 0$
    (b) $\langle\uparrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = \hbar/2$
    (c) $\langle\uparrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = \hbar$
    (d) $\langle\uparrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = 3\hbar/2$

40. *Which one of the following is true?*

(a) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = 0$

(b) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = \hbar/2$

(c) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = \hbar$

(d) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\uparrow\uparrow\uparrow\rangle = 3\hbar/2$

41. *Which one of the following is true?*

(a) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\downarrow\uparrow\uparrow\rangle = 0$

(b) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\downarrow\uparrow\uparrow\rangle = \hbar/2$

(c) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\downarrow\uparrow\uparrow\rangle = \hbar$

(d) $\langle\downarrow\uparrow\uparrow|\hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}|\downarrow\uparrow\uparrow\rangle = 3\hbar/2$

42. *Construct the matrix representation of the operator* $\hat{S}_z = \hat{S}_{1z}+\hat{S}_{2z}+\hat{S}_{3z}$ *in the uncoupled representation. Explicitly show the order in which you chose the basis vectors to construct the matrix.*

The goal of this part of the tutorial is to review the basics of

- two spin-1/2 systems in the coupled representation

At the end of this part of the tutorial, you should be able to do problems of the type given in the pretest. It is important that you work through this tutorial with a blank sheet of paper and pen so that you can write down important ideas you learned as you go along. It is recommended that you work through the tutorial on a single spin system before working on this tutorial.

Basis Vectors in the "Coupled" Representation

• We will now develop a method for representing the basis vectors for a multi-spin system in a representation often called the "coupled" representation.

• In this representation, it is NOT possible to write the basis vectors for a two spin-system in terms of products of basis vectors for single spin-systems (unlike the uncoupled representation where for two spin-1/2 systems $|\uparrow\downarrow\rangle \equiv |\uparrow\rangle_1 |\downarrow\rangle_2$).

• We will then write the Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ in the pretest in the "coupled" representation.

1. The spin quantum numbers for each spin in a two spin-1/2 systems are $s_1 = 1/2$ and $s_2 = 1/2$. List all the possible total spin quantum numbers where the total spin angular momentum is defined as $\vec{S} = \vec{S}_1 + \vec{S}_2$.
   (a) 1
   (b) 0
   (c) 1 and 0
   (d) any positive integer value

2. If the total spin quantum number for a system is $s = 1$, list all the possible m_z, the quantum number corresponding to the z component of the total spin.
   (a) 1, 0 and -1
   (b) 1, -1
   (c) 1/2, -1/2
   (d) 1, 0

3. If the total spin quantum number for a system is $s = 0$, list all the possible m_z, the quantum number corresponding to the z component of the total spin.
   (a) 0
   (b) 0 and 1
   (c) 1 and -1
   (d) 1

Notation for two spin systems in the coupled representation:

In the "coupled" representation, the basis vectors are the simultaneous eigenstates of $\hat{S}^2, \hat{S}_z, \hat{S}_1^2$, and $\hat{S}_2^2$. (where $\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2$ is the square of the magnitude of the total angular momentum operator and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$ is the z component of the total angular momentum operator).

For each spin, the spin quantum number is fixed (e.g., $s_1 = 1/2$ and $s_2 = 1/2$ for each spin-1/2 system), so we will label the orthonormal basis vectors only in terms of the total spin quantum number S and the z component of the total spin quantum number m_z in the compact notation: $|S, s_1, s_2, m_z\rangle = |S, m_z\rangle$.

The eigenvalue equations for the operators $\hat{S}^2, \hat{S}_z, \hat{S}_1^2, \hat{S}_2^2$ are:

$\hat{S}^2|S, m_z\rangle = \hbar^2 S(S+1)|S, m_z\rangle$

$\hat{S}_1^2|S, m_z\rangle = \hbar^2 s_1(s_1+1)|S, m_z\rangle$

$\hat{S}_2^2|S, m_z\rangle = \hbar^2 s_2(s_2+1)|S, m_z\rangle$

$\hat{S}_z|S, m_z\rangle = \hbar m_z|S, m_z\rangle)$

4. Which one of the following sets are the basis vectors for the two spin-1/2 systems in the "coupled" representation in which the basis vectors are simultaneous eigenstates of $\hat{S}^2, \hat{S}_z, \hat{S}_1^2, \hat{S}_2^2$?
   (a) $|1/2, 0\rangle$, $|0, 1/2\rangle$, $|0, 0\rangle$, $|0, 1\rangle$
   (b) $|1, 1\rangle$, $|1, 0\rangle$, $|1, -1\rangle$, $|0, 0\rangle$
   (c) $|1, 0\rangle$, $|0, 0\rangle$, $|0, 1\rangle$
   (d) $|1, -1\rangle$, $|1, 1\rangle$, $|1, 0\rangle$

5. Which one of the following statements is correct for two spin-1/2 systems with $s_1 = 1/2$ and $s_2 = 1/2$ in the coupled state $|1, 1\rangle$ in the coupled representation?
   (a) $\hat{S}^2|1,1\rangle = 2\hbar^2|1,1\rangle, \hat{S}_1^2|1,1\rangle = 3\hbar^2/4|1,1\rangle, \hat{S}_2^2|1,1\rangle = 3\hbar^2/4|1,1\rangle, \hat{S}_z|1,1\rangle = \hbar|1,1\rangle$
   (b) $\hat{S}^2|1,1\rangle = 2\hbar^2|1,1\rangle, \hat{S}_1^2|1,1\rangle = \hbar^2|1,1\rangle, \hat{S}_2^2|1,1\rangle = \hbar^2|1,1\rangle, \hat{S}_z|1,1\rangle = \hbar|1,1\rangle$
   (c) $\hat{S}^2|1,1\rangle = 2\hbar^2|1,1\rangle, \hat{S}_1^2|1,1\rangle = 2\hbar^2|1,1\rangle, \hat{S}_2^2|1,1\rangle = 2\hbar^2|1,1\rangle, \hat{S}_z|1,1\rangle = \hbar|1,1\rangle$
   (d) $\hat{S}^2|1,1\rangle = \hbar^2|1,1\rangle, \hat{S}_1^2|1,1\rangle = 3\hbar^2/2|1,1\rangle, \hat{S}_2^2|1,1\rangle = 3\hbar^2/2|1,1\rangle, \hat{S}_z|1,1\rangle = \hbar|1,1\rangle$

6. Which one of the following statements is correct for two spin-1/2 systems with $s_1 = 1/2$ and $s_2 = 1/2$ in the coupled state $|1, 0\rangle$ in the coupled representation?
   (a) $\hat{S}^2|1,0\rangle = 2\hbar^2|1,0\rangle, \hat{S}_1^2|1,0\rangle = (3\hbar^2/4)|1,0\rangle, \hat{S}_2^2|1,0\rangle = (3\hbar^2/4)|1,0\rangle, \hat{S}_z|1,0\rangle = 0|1,0\rangle$
   (b) $\hat{S}^2|1,0\rangle = 2\hbar^2|1,0\rangle, \hat{S}_1^2|1,0\rangle = \hbar^2|1,0\rangle, \hat{S}_2^2|1,0\rangle = \hbar^2|1,0\rangle, \hat{S}_z|1,0\rangle = \hbar|1,0\rangle$
   (c) $\hat{S}^2|1,0\rangle = 2\hbar^2|1,0\rangle, \hat{S}_1^2|1,0\rangle = 2\hbar^2|1,0\rangle, \hat{S}_2^2|1,0\rangle = 2\hbar^2|1,0\rangle, \hat{S}_z|1,0\rangle = 0|1,0\rangle$
   (d) $\hat{S}^2|1,0\rangle = \hbar^2|1,0\rangle, \hat{S}_1^2|1,0\rangle = 0|1,0\rangle, \hat{S}_2^2|1,0\rangle = 0|1,0\rangle, \hat{S}_z|1,0\rangle = 0|1,0\rangle$

7. Consider the basis vectors in the previous problem for two spin-1/2 systems in the "coupled" representation. Which one of the following is correct about their scalar (inner) product?

   (a) $\langle 1, -1|1, 1\rangle = 1$

   (b) $\langle 1, 0|1, 1\rangle = 0$

   (c) $\langle 0, 0|1, 1\rangle = 1$

   (d) $\langle 0, 0|0, 0\rangle = 0$

8. Choose all of the following statements that are correct about the differences between the "coupled" and "uncoupled" representations of the multi-spin system.

   (I) Working entirely within the coupled representation, you cannot decompose the product state of a two-spin system into products of states of each individual spin.

   (II) Working entirely within the uncoupled representation, you can decompose the product state of a two-spin system into products of states of each individual spin.

   (III) The basis vectors in the uncoupled representation are eigenstates of $\hat{S}_1^2$, $\hat{S}_{1z}$, $\hat{S}_2^2$, and $\hat{S}_{2z}$, whereas the basis vectors in the coupled representation are eigenstates of $\hat{S}^2$, $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$, $\hat{S}_1^2$ and $\hat{S}_2^2$.

   (a) (I) and (II) only

   (b) (I) and (III) only

   (c) (II) and (III) only

   (d) (I), (II) and (III).

9. Choose all of the following statements that are correct:

   (I) Basis vectors in both the uncoupled and coupled representations are eigenstates of $\hat{S}_1^2$, and $\hat{S}_2^2$.

   (II) Basis vectors in both the uncoupled and coupled representations are eigenstates of $\hat{S}_{1z}$.

   (III) Basis vectors in both the uncoupled and coupled representations are the eigenstates of $\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2$.

   (a) (I) only

   (b) (I) and (II) only

   (c) (I) and (III) only

   (d) (I), (II) and (III).

Constructing $\hat{H}_1$ and $\hat{H}_2$ matrices for two spin-1/2 systems in the "coupled" representation

10. Consider the basis vector $|1,1\rangle$ for two spin-1/2 systems in the "coupled" representation. Which one of the following is the correct expression for $(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|1,1\rangle$?
   (a) $\hbar^2/2$
   (b) $(\hbar^2/2)|1,1\rangle$
   (c) $-2\hbar^2$
   (d) $(-2\hbar^2)|1,1\rangle$

11. Consider the basis vector $|1,1\rangle$ for two spin-1/2 systems in the "coupled" representation. Which one of the following is the correct expression for $\langle 1,1|(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|1,1\rangle$?
   (a) $\hbar^2/2$
   (b) $-\hbar^2/2$
   (c) $2\hbar^2$
   (d) $-2\hbar^2$

12. Consider the basis vectors for two spin-1/2 systems in the "coupled" representation. Which one of the following is the correct expression for $\langle 1,0|(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|1,1\rangle$?
   (a) $\hbar^2/2$
   (b) $2\hbar^2$
   (c) $-2\hbar^2$
   (d) 0

13. Consider the basis vector $|0,0\rangle$ for two spin-1/2 systems in the "coupled" representation. Which one of the following is the correct expression for $(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|0,0\rangle$?
   (a) 0
   (b) $(-\hbar^2/2)|0,0\rangle$
   (c) $-2\hbar^2$
   (d) $(-3\hbar^2/2)|0,0\rangle$

14. Consider the basis vector $|0,0\rangle$ for two spin-1/2 systems in the "coupled" representation. Which one of the following is the correct expression for $\langle 00|(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|0,0\rangle$?
   (a) 0
   (b) $-2\hbar^2$
   (c) $-3\hbar^2$
   (d) $-3\hbar^2/2$

15. Consider the basis vectors for two spin-1/2 systems in the "coupled" representation. Which one of the following is the correct expression for $\langle 1,0|(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|0,0\rangle$?

(a) $\hbar^2/2$

(b) $2\hbar^2$

(c) $-2\hbar^2$

(d) 0

16. Consider the following conversation between Pria and Mira:

- Pria: In the "coupled" representation, $\hat{H}_1$ can be written as a diagonal matrix.
- Mira: Yes. The basis vectors $|S, m_z\rangle$ are eigenstates of $\hat{H}_1$ because $\hat{H}_1$ involves $\hat{S}^2$, $\hat{S}_1^2$, and $\hat{S}_2^2$. Therefore, $\hat{H}_1|S, m_z\rangle$ will give a "number" multiplied by $|S, m_z\rangle$ and when you calculate the matrix elements by taking scalar products, orthogonality of states will make only the diagonal terms non-zero.

Do you agree with Pria and Mira?

(a) Yes.

(b) No.

17. Which one of the following is the correct expression for $\hat{H}_1 = 2E_0/\hbar^2(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)$ in the "coupled" representation? Choose the basis vectors in the order $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, $|0,0\rangle$ for calculating $\hat{H}_1$ in the matrix form.

(a) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & E_0 & 0 \\ 0 & 0 & 0 & -3E_0 \end{pmatrix}$

(b) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & E_0 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix}$

(d) $\begin{pmatrix} E_0 & 0 & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & E_0 & 0 \\ 0 & 0 & 0 & -2E_0 \end{pmatrix}$

Note: Recall that earlier when you calculated $\hat{H}_1$ in the "uncoupled representation" (in which the basis vectors are eigenstates of $\hat{S}_1^2$, $\hat{S}_{1z}$, $\hat{S}_2^2$, and $\hat{S}_{2z}$), $\hat{H}_1$ had non-diagonal matrix elements because $\hat{S}^2$ in $\hat{H}_1$ does not commute with $\hat{S}_{1z}$ or $\hat{S}_{2z}$. Now in the "coupled representation" for which the basis vectors are eigenstates of $\hat{S}^2$, $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$, $\hat{S}_1^2$ and $\hat{S}_2^2$, $\hat{H}_1$ is diagonal because $\hat{H}_1$ commutes with the operators $\hat{S}^2$, $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$, $\hat{S}_1^2$ and $\hat{S}_2^2$. This example shows that the choice of basis or co-ordinate system is important because a convenient choice of basis can put the operators corresponding to physical observable in the diagonal form.

18. Which one of the following is the correct matrix representation for $\hat{H}_2 = -\mu(\hat{\vec{S}}_1 \cdot \vec{B} + \hat{\vec{S}}_2 \cdot \vec{B})$ in the "coupled" representation where the magnetic field B is in the z direction? Express the matrix in the block diagonal form if possible (where all the non-zero terms are confined to a smaller block than 4×4) by writing the basis vectors in suitable order when expressing the matrix.
Hint: $\hat{H}_2 = -\mu B(\hat{S}_{1z} + \hat{S}_{2z}) = -\mu B\hat{S}_z$

(a) $-\mu B\hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(b) $-\mu B\hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $-\mu B\hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $-\mu B\hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Summary: For the pretest problem, the "coupled" representation is more convenient for writing $\hat{H}_1$ and $\hat{H}_2$ because both $\hat{H}_1$ and $\hat{H}_2$ can be represented as diagonal matrices in this basis. In the "uncoupled" representation, $\hat{H}_2$ can be written as a diagonal matrix but not $\hat{H}_1$.

Generalizing to product space of other spin systems

1. What is the dimensionality of the spin-space of one spin-one system?
   (a) 2
   (b) 3
   (c) 4
   (d) 6

2. If the total spin quantum number for a system is $S = 1$, list all the possible values of m_z, the quantum number corresponding to the z component of the total spin.
   (a) 1
   (b) 1, 0
   (c) 1, 0, -1
   (d) 1/2, -1/2

3. What is the dimensionality of the spin-space of two spin-one systems?
   (a) 3
   (b) 4
   (c) 6
   (d) 9

4. Which one of the following is a complete set of basis vectors $|m_{1z}, m_{2z}\rangle$ for the product space of two spin-one systems in the "uncoupled basis"? Note: Although for spin half particles, we used $\uparrow$ for $m_z = 1/2$ and $\downarrow$ for $m_z = -1/2$, we will use the usual quantum numbers $(1, 0, -1)$ to denote the states of a spin-one system.
   (a) $|1\rangle$, $|0\rangle$, $|-1\rangle$
   (b) $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$
   (c) $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, $|0,0\rangle$, $|0,-1\rangle$, $|-1,-1\rangle$
   (d) $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, $|0,1\rangle$, $|0,0\rangle$, $|0,-1\rangle$, $|-1,1\rangle$, $|-1,0\rangle$, $|-1,-1\rangle$

5. Write down the operator $\hat{\vec{S}} = \hat{\vec{S_1}} + \hat{\vec{S_2}}$ as a 9×9 matrix for the product space of two spin-one systems in the "uncoupled basis"? You can use the complete set of basis vectors $|m_{1z}, m_{2z}\rangle$ in the previous question to construct the matrix. Show explicitly the order in which use chose the basis vectors to construct the 9×9 matrix for the operator $\hat{\vec{S}} = \hat{\vec{S_1}} + \hat{\vec{S_2}}$.

6. How will the operator matrix for $\hat{\vec{S_1}} + \hat{\vec{S_2}}$ that you constructed above differ from the operator matrix for $\hat{\vec{S_1}} + \hat{\vec{S_2}}/2$ for the product space of two spin-one systems in the "uncoupled basis"? In particular, are both $\hat{\vec{S_1}} + \hat{\vec{S_2}}$ and $\hat{\vec{S_1}} + \hat{\vec{S_2}}/2$ diagonal matrices in the uncoupled representation? Explain.

7. If instead of choosing the uncoupled representation to construct the basis vectors for the product space of two spin-one systems, we choose the coupled representation, will the dimensionality of the product space still be same (in particular, 9)? Explain.

8. To denote the basis vectors for the product space of two spin-one systems in the "coupled" basis, we need the total spin quantum numbers corresponding to $S_1 = 1$ and $S_2 = 1$. List all of the possible total spin quantum numbers for the product space where $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$?
   (a) 2, -2, 1, -1 and 0
   (b) 2, -2 and 0
   (c) 2 and 0
   (d) 2, 1 and 0

9. If the total spin quantum number for the system is $S = 2$, list all the possible values of m_z, the quantum number corresponding to the z component of the total spin.
   (a) 2,0
   (b) 2,1,0
   (c) 2, 0, -2
   (d) 2,1,0,-1,-2

10. Which one of the following is a complete set of basis vectors $|S, m_z\rangle$ for the product space of two spin-one systems in the "coupled basis" (where $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ and $m_z = m_{1z} + m_{2z}$?
   (a) $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$
   (b) $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$
   (c) $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$, $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$
   (d) $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$, $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, $|00\rangle$

11. For two spin-one systems, one basis vector in the "coupled" basis is $|S, m_z\rangle = |2,2\rangle$. Which one of the following statements is false about this state (only one statement is false, three of the statements are correct)?
   (a) $\hat{S}^2|2,2\rangle = \hbar^2 2(2+1)|2,2\rangle = 6\hbar^2|2,2\rangle$
   (b) $\hat{S}_z|2,2\rangle = 4\hbar|2,2\rangle$
   (c) Since $|S, m_z\rangle$ is a short notation for $|S, S_1, S_2, m_z\rangle$, we can write $|2,2\rangle = |2,1,1,2\rangle$
   (d) $\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2$ and $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$

12. Write down the operator $\hat{\vec{S}} = \hat{\vec{S_1}} + \hat{\vec{S_2}}$ as a 9×9 matrix for the product space of two spin-one systems in the "coupled basis"? You can use the complete set of basis vectors $|S, m_z\rangle$ you selected in a previous question to construct the matrix. Show explicitly the order in which use chose the basis vectors $|S, m_z\rangle$ to construct the 9×9 matrix for the operator $\hat{\vec{S}} = \hat{\vec{S_1}} + \hat{\vec{S_2}}$.

13. How will the operator matrix for $\hat{\vec{S_1}} + \hat{\vec{S_2}}$ that you constructed above differ from the operator matrix for $\hat{\vec{S_1}} + \hat{\vec{S_2}}/2$ for the product space of two spin-one systems in the "coupled basis"? In particular, are both $\hat{\vec{S_1}} + \hat{\vec{S_2}}$ and $\hat{\vec{S_1}} + \hat{\vec{S_2}}/2$ diagonal matrices in the coupled representation? Explain. You need not calculate the matrix elements of $\hat{\vec{S_1}} + \hat{\vec{S_2}}/2$ in the coupled representation to make your point.

14. If the total spin quantum number for a system is $S = 3/2$, what is the dimensionality of the corresponding spin-space?
(a) 2
(b) 3
(c) 4
(d) 5

15. If the total spin quantum number for a system is $S = 3/2$, list all the possible values of m_z, the quantum number corresponding to the z component of the total spin.
(a) 3/2, 0, -3/2
(b) -3/2, -1/2, 3/2
(c) -3/2, 1/2, 3/2
(d) -3/2, -1/2, 1/2, 3/2

16. What is the dimensionality of the product space of a spin-one and a spin-3/2 systems? Explain.

17. Write down a complete set of basis vectors $|m_{1z}, m_{2z}\rangle$ for the product space of a spin-one and a spin-3/2 systems in the "uncoupled basis".

18. If instead of choosing the uncoupled representation to construct the basis vectors for the product space of a spin-one and a spin-3/2 systems, we choose the coupled representation, will the dimensionality of the product space still be same (in particular, 12)? Explain.

19. To denote the basis vectors for the product space of a spin-one and a spin-3/2 systems in the "coupled" basis, we need the total spin quantum numbers corresponding to $S_1 = 1$ and $S_2 = 3/2$. List all of the possible total spin quantum numbers for the product space for a spin-one and a spin-3/2 systems where $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$?
    (a) 5/2, -5/2, 3/2, -3/2, 1/2 and -1/2
    (b) 5/2, -5/2, 3/2 and -3/2
    (c) 5/2 and 1/2
    (d) 5/2, 3/2 and 1/2

20. If the total spin quantum number for a system is $S = 5/2$, list all the possible values of m_z, the quantum number corresponding to the z component of the total spin.
    (a) -5/2, -3/2, 3/2, 5/2
    (b) -5/2, -3/2, 0, 3/2, 5/2
    (c) -5/2, -3/2, -1/2, 0, 1/2, 3/2, 5/2
    (d) -5/2, -3/2, -1/2, 1/2, 3/2, 5/2

21. Write down a complete set of basis vectors $|S, m_z\rangle$ for the product space of a spin-one and a spin-3/2 systems in the "coupled basis" (where $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ and $m_z = m_{1z} + m_{2z}$?

22. What is the dimensionality of the vector space for n spin one half systems?
    (a) $(1/2)^n$
    (b) $n^{1/2}$
    (c) n^2
    (d) 2^n

23. What is the dimensionality of the vector space for n spin m systems?
    (a) m^n
    (b) n^m
    (c) $(2m+1)^n$
    (d) $(2n+1)^m$

Product space of systems involving any angular momentum (spin or orbital)

Consider the following conversation between Pria and Mira:

• Pria: Will the addition of angular momentum formalism we learned so far for the product space of "spin" angular momenta apply to the product space of systems involving orbital angular momenta or to the product space of systems involving an orbital angular momentum and a spin angular momentum?

• Mira: Yes, the formalism of the addition of angular momentum is applicable to both orbital and spin angular momenta so we can find the product space of systems involving any angular momentum regardless of whether it is spin or orbital. However, the orbital angular momentum quantum number can only take integer values whereas the spin quantum number can take both integer and half-integer values.

• Pria: I see. So the example of the product space formalism we worked on earlier for two spin-one systems may as well have been for two orbital angular momentum quantum number one systems?

• Mira: Yes. Similarly, the example of the product space formalism we worked on earlier for a spin-one and a spin-3/2 systems may as well have been for an orbital angular momentum quantum number one and a spin-3/2 systems.

Do you agree with Mira? Explain.