Part 1 Quantum measurement without time evolution after measurement

Note:

- ✧ In the simulations, the y-axis represents $|\psi(x)|$ (the absolute value of the wavefunction) instead of $\psi(x)$.

<u>What is quantum measurement?</u>

No matter what the initial state of the quantum system is, when we measure an observable, the system collapses into an eigenstate of the corresponding operator. Therefore, measurement of an observable can be considered as projecting the initial state onto an eigenstate of the operator. For example, suppose we measure the energy of a particle in the initial state $|\Psi\rangle$ which is not an energy eigenstate. Let the energy eigenstates (eigenstates of the Hamiltonian) be denoted in order of increasing energy as $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$, ..., $|\psi_n\rangle$, …, where n is a positive integer. Then, to find the probability of measuring energy E_n, we can project the initial state $|\Psi\rangle$ onto the energy eigenstate $|\psi_n\rangle$ as $\langle\psi_n|\Psi\rangle$ and then calculate the probability as $|\langle\psi_n|\Psi\rangle|^2$.

Now answer the following questions.

(1) Write $\langle\psi_n|\Psi\rangle$ in the position representation? (Hint : Spectral decomposition of identity gives $\int dx|x\rangle\langle x| = 1$)

(2) What is the dimension /unit of $\langle\psi_n|\Psi\rangle$?

A. Length, e.g., nanometer (nm)
B. Inverse length, e.g., $1/nm$
C. Inverse square length, e.g., $1/nm^2$
D. Dimensionless / Unitless

(3) What is the physical meaning of $|\langle\psi_n|\Psi\rangle|^2$?

Now let's use the idea of projecting a general state along an energy eigenstate to find the probability of measuring a particular energy for a 1-D infinite square well.

1-D infinite square well

For a particle in a 1-D infinite square well with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ $(V(x) = 0$ when $0 < x < a$ and $V(x) = +\infty$ otherwise), the n^{th} energy is $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ (n=1,2,3,…), and the energy eigenfunction corresponding to E_n is $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ when $0 < x < a$ and $\psi_n(x) = 0$ elsewhere. Answer the following questions. (Questions 1--10)

1. Suppose the initial state of the particle is $|\psi_1\rangle$. If we measure the energy of the particle, what result(s) can we obtain?

A. Only E_1

B. Any of E_n, n=1,2,3,…

C. $\sum c_n E_n$, c_n are constants and at least two of c_n are non-zero, n=1,2,3,…

D. Any value of energy E is possible as long as $E \geq E_1$

2. In the previous problem (*problem 1*), after the measurement of energy, what state will the particle be in?

A. Definitely in the state $|\psi_1\rangle$

B. Any of the states $|\psi_n\rangle$, n=1,2,3, …

C. $\sum_n A_n|\psi_n\rangle$, A_n are constants and at least two of A_n are non-zero, n=1,2,3,…

D. None of the above

Simulation 1

Double click the simulation “psi1” on the left column of the program window. The initial state of the system in this simulation is $|\psi_1\rangle$. Next, choose “*E*” (energy) at the lower right corner of the new window. Click the button “measure” in the lower middle part of the window. Does the shape of the absolute value of the wave function change? Is this result consistent with your answer to *question 2*? What is the measured energy corresponding to the wave function you have obtained?

3. Suppose the initial state of the particle is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. If we measure the energy of the particle, what result can we obtain?

A. $(E_1 + E_2)/2$

B. E_1 or E_2

C. Any of E_n, n=1,2,3,…

D. $\sum c_n E_n$, c_n are constants and at least two of c_n are non-zero, n=1,2,3,…

4. In the previous problem (*problem 3*), after the measurement of energy, what state will the particle be in?

A. $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$.

B. Only $|\psi_1\rangle$ or $|\psi_2\rangle$

C. Any of $|\psi_n\rangle$ with non-zero probability, n=1,2,3,…

D. $\sum_n A_n |\psi_n\rangle$, A_n are constants and at least two of A_n are non-zero, n=1,2,3,…

E. None of the above

5. Suppose the initial state is $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. If you measure the energy of the system, what is the probability of measuring energy E_n in Dirac notation? For the given initial state, the probability of measuring which of the energies is non-zero? Is this result consistent with your answers to *question 3*?

Simulation 2

Choose the simulation "psi1+psi2". The initial state of the system in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Next, choose "*E*" (energy) at the lower right corner of the window. Then click the button "measure" in the lower middle of the window. Does the shape of the absolute value of the wave function change?

Now click the button with a curved arrow (just to the left of the measure button) to reset the initial state to $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Then measure the energy again. Do you obtain the same state after this second measurement of energy as what you obtained after the first measurement of energy in the state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$? If yes, do you expect that you *may* obtain a different state when you measure energy in the next trial after resetting the initial state to $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$? Is this result consistent with your answer to question 4?

Since $|\psi_1\rangle$ *and* $|\psi_2\rangle$ *are orthogonal (* $\langle\psi_1|\psi_2\rangle = 0$ *),* $\langle\psi_1|\left(\frac{1}{\sqrt{2}}|\psi_1 + \psi_2\rangle\right) = \frac{1}{\sqrt{2}}\langle\psi_1|\psi_1\rangle$ *and the probability of measuring* E_1 and *the initial state collapsing into* ψ_1 *after the measurement of energy is* $\left|\frac{1}{\sqrt{2}}\langle\psi_1|\psi_1\rangle\right|^2 = \frac{1}{2}$. *Similarly, the probability of measuring* E_2 *and collapsing the initial state into* ψ_2 *after the measurement of energy is* $\left|\frac{1}{\sqrt{2}}\langle\psi_2|\psi_1 + \psi_2\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\langle\psi_2|\psi_2\rangle\right|^2 = \frac{1}{2}$. *For any other energy eigenstate* ψ_n *,* $\langle\psi_n|\psi_1 + \psi_2\rangle = 0$, *so the probability is zero for those states and the system cannot collapse to any* $|\psi_n\rangle$ *other than* $|\psi_1\rangle$ *or* $|\psi_2\rangle$ *when we measure the energy for the state* $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$.

Now use the method of projecting the general state along an energy eigenstate to answer the following questions (questions 6-8).

6. Suppose the normalized initial state of the particle is $\sum_n A_n|\psi_n\rangle$, where A_n are constants and at least two of A_n are non-zero (n=1, 2, 3, …). If we measure the energy of the particle, what result can we obtain?

A. Any of E_n for which $A_n \neq 0$.

B. $A_n E_n$

C. $\sum_n A_n E_n$

D. $\sum_n |A_n|^2 E_n$

7. In the previous problem (*question 6*), what is the probability of measuring energy E_n in the state $\sum_n A_n|\psi_n\rangle$? Note that A_n can be a complex number.

A. A_n

B. $|A_n|$

C. $(A_n)^2$

D. $|A_n|^2$

8. In *problem 6*, after the measurement of energy, what normalized state will the particle be in?

A. Any one of the energy eigenstates $|\psi_n\rangle$ corresponding to the energy measured.

B. Any one of the states $A_n|\psi_n\rangle$.

C. $\sum_n A_n|\psi_n\rangle$

D. $\sum_n |A_n|^2|\psi_n\rangle$

E. None of the above

Simulation 3

Choose the simulation "psi1+psi_n". The initial state of the system in this simulation is $\sum_n A_n|\psi_n\rangle$ with equal coefficient A_n for $n \le 9$ and $A_n = 0$ for $n > 9$. Next, choose "*E*" (energy) at the lower right corner of the window. Then click the button "measure" in the lower middle of the window. What state do you obtain? Set back the simulation to the initial state and measure again to check whether you can get a different state. Explain what is the probability of obtaining a particular state $|\psi_n\rangle$.

9. The orthonormal energy eigenfunctions ψ_n for a 1D infinite square well satisfy $\int_{-\infty}^{+\infty} \psi_n^*(x)\psi_m(x)dx = \delta_{mn}$, where $\delta_{mn} = 1$ when *m*=*n*, and $\delta_{mn} = 0$ otherwise. Any state $|\Psi\rangle$ can be expressed as $|\Psi\rangle = \sum_n A_n|\psi_n\rangle$ because $|\psi_n\rangle$ form a complete set of vectors for the Hilbert space in which the state of the system lies. Find A_n in terms of $|\Psi\rangle$ and $|\psi_n\rangle$ first in the Dirac notation form and then in the integral form in the position representation. (*The hint is on the last page of part 1, after question 25.*)

10. Suppose the wavefunction of the particle in the initial state is $\Psi(x) = Ax(a-x)$ (*A* is a normalization constant) when $0 < x < a$ and $\Psi(x) = 0$ otherwise. If we measure the energy of the particle, what is the probability of obtaining E_n? (*n*=1,2,3,…) Use the idea of projecting the initial state along an energy eigenstate to find the probability of measuring energy E_n . Write down your answer in both the Dirac notation and integral form in the position representation You need NOT evaluate the integral but you should show suitable limits for the integral.

For a particle interacting with a simple harmonic oscillator (SHO) potential energy, the energies are $E_n = (n+\frac{1}{2})\hbar\omega$ *(n=0,1,2,...), and the energy eigenfunctions corresponding to* E_n *are* $\psi_n(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$, *where* $H_n(\xi)$ *is the n*th *Hermite polynomial and* $\xi \equiv \sqrt{\frac{m\omega}{\hbar}} x$ *is a dimensionless variable. The first three Hermite polynomials are* $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$. *Answer questions 11 & 12.*

11. Suppose the wavefunction of a simple harmonic oscillator in the initial state is a Gaussian function $\psi(\xi) = Ae^{-\xi^2/2}$, where A is a normalization constant. If we measure the energy of the simple harmonic oscillator, what energy can we obtain?

A. $E = \hbar\omega$ only

B. $E = \sum_{n=0}^{\infty}(n+\frac{1}{2})\hbar\omega$

C. $E = \frac{1}{2}\hbar\omega$ only

D. Any of the energies $E_n = (n+\frac{1}{2})\hbar\omega$, n=0,1,2,...

12. Suppose the initial state of a simple harmonic oscillator is a Gaussian function not centered around $x = 0$ (where the potential energy is minimum). The initial state can be expressed as $\psi(\xi) = Ae^{-(\xi-\xi_0)^2/2}$, where A is a normalization constant and $\xi_0 \neq 0$. If we measure the energy of the simple harmonic oscillator, what result(s) can we obtain?

A. $E = \hbar\omega$ only

B. $E = \sum_{n=0}^{\infty}(n+\frac{1}{2})\hbar\omega$

C. Only ground state energy $E_0 = \frac{1}{2}\hbar\omega$ since the wavefunction is still Gaussian

D. Any of the energies $E_n = (n+\frac{1}{2})\hbar\omega$, n=0,1,2,...

No matter what the initial state is, when we measure the energy of a quantum SHO, we always measure an energy eigenvalue (allowed energy) and collapse the wavefunction into an energy eigenstate of the SHO. It is the Hamiltonian of the system that determines the energy eigenstates and allowed energies of the system. The initial state determines the possibility of collapsing into different energy eigenstates and measuring the corresponding energy when measuring the energy of the system.

Measurement of the position when the initial state is an energy eigenstate,

Consider an electron in a 1-D infinite square well with $V = 0$ when $0 < x < a$ and $V = +\infty$ otherwise. Answer the following questions (*questions 13 -- 18*).

13. Suppose the initial state of the particle is the ground state $|\psi_1\rangle$. If we measure the position of the particle, what possible values can we obtain? Will we obtain the same value if we perform position measurements on a large number of identically prepared systems? Explain.

A. $x = a$ only

B. $x = a/2$ only

C. $0 < x < a$

D. Any value between $-\infty$ and $+\infty$

14. In the previous problem (*question 13*), after the measurement of position, which one of the following wavefunctions will the particle be in if we find the particle at $x = x_0$?

A. $\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

B. $\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi(x - x_0)}{a}\right)$

C. $\psi(x) = \delta(x)$

D. $\psi(x) = \delta(x - x_0)$

15. Let's find the probability density of measuring the position of the particle in state $|\psi_1\rangle$ (*questions 13&14*) using the projection method in the Dirac notation and in the position representation. First write down the wavefunction (in the position representation) of the particle in the initial state $|\psi_1\rangle$ (the ground state). Then, consider the wavefunction of the position eigenstate $|x_0\rangle$ with eigenvalue x_0. Calculate the projection $\langle x_0 | \psi_1 \rangle$ of the state $|\psi_1\rangle$ along the position eigenstate $|x_0\rangle$ in the position represenation by writing down the integral explicitly. What is the probability density $|\langle x_0 | \psi_1 \rangle|^2$ for finding the particle at the position $x = x_0$? Is this result consistent with Born's interpretation of the wavefunction? Explain. (Hint: The spectral decomposition of identity is $\int_{all} dx' |x'\rangle\langle x'| = 1$.)

16. Born's statistical interpretation of the wavefunction says that $|\psi(x,t)|^2 dx$ gives the probability of finding the particle between *x* and *x*+*dx* at time *t*. Does your result in *question 15* support this statistical interpretation? Explain.

Simulation 4

Double click the simulation "QM measurement". Then choose the simulation "psi1". The initial state in this simulation is $|\psi_1\rangle$. Next, choose "*x*" (position) and click the button "measure" in the lower middle of the window. What is the (approximate) position of the particle measured? Set back the simulation to the initial state $|\psi_1\rangle$ and measure the position again. Is the particle found at the same position as your first measurement? Explain your observation. Is this result consistent with your answer to *question 13*?

(Note that the position eigenfunction in the simulation is not a perfect delta function due to constraints in the simulation. However, the delta function is an ideal model which does not exist in the real world. For example, when an electron in a double slit experiment hits the far away screen, it leaves a spot with a finite width.)

17. Suppose the initial state of the particle is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. If we measure the position of the particle, what result can we obtain?

A. $x = a$ only

B. $x = a/2$ only

C. $0 \le x \le a$

D. Any value between $-\infty$ and $+\infty$

18. In the previous problem (*question 17*), after the measurement of position, what state will the particle be in if we find the particle at $x = x_0$? Write down this state in Dirac notation and in position representation. What is the probability density for measuring the position $x = x_0$? (Hint: $\int_{all} dx |x\rangle\langle x| = 1$. You can calculate the projection $\langle x_0 | \Psi\rangle$ in the position representation by writing down the integral $\int_{all} dx \langle x_0 | x\rangle\langle x | \Psi\rangle = \int_{all} dx \delta(x - x_0)\Psi(x)$.)

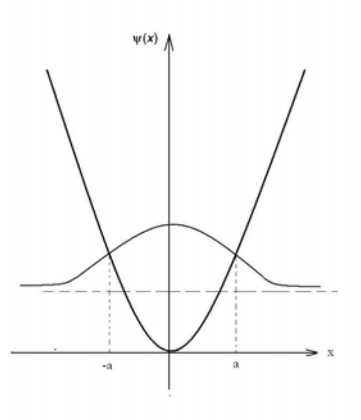
Simulation 5

Double click the simulation "QM measurement". Then choose the simulation "psi1+psi2". The initial state of the system in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Next, choose "x" (position) and click the button "measure" in the lower middle of the window. What is the (approximate) position of the particle? Set back the simulation to the initial state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ and measure the position again. Is the particle found at the same position as your first measurement? Explain your observation. Is this result consistent with your answer to *question 17*?

Measurement of the position when the initial state is an energy eigenstate of the SHO,

Consider a particle interacting with a simple harmonic oscillator potential energy well. Answer questions 19 & 20.

19. A simple harmonic oscillator is in the ground state with a normalized Gaussian wave function as shown. If we measure the position of the particle, what results can we obtain? The classical turning points are $\pm a$ where $a = \sqrt{\frac{\hbar}{m\omega}}$



A. $x = 0$ only
B. $x = \pm a$ only
C. Any value between $-a$ and a
D. Any value between $-\infty$ and $+\infty$

20. In the previous problem (*question 19*), after the measurement of position, what state will the particle be in if we find the particle at $x = x_0$? Write down this state in position representation. Use the idea of projection to write the probability density of measuring $x = x_0$ in Dirac notation and in the position representation when the position measurement was performed in the ground state of the SHO.

Measurement of the position when the initial state is arbitrary

Consider a particle in a 1-D infinite square well with $V = 0$ when $0 < x < a$ and $V = +\infty$ elsewhere. Answer the following questions (21 & 22).

21. Suppose the wavefunction of a particle in the initial state is $\Psi(x) = A\sin^2(\pi x / a)$ where A is a normalization constant. If we measure the position of the particle, what is the probability density for finding the particle at $x = x_0$? Use the idea of projection to explain your answer by writing down the probability density in Dirac notation and in the position representation.

22. In the previous problem (*question 21*), immediately after the measurement of position, what state will the particle be in? Write down the wavefunction of the particle in this state mathematically and also sketch it graphically in the position representation.

23. Choose all of the following statements that are correct.
    (1) The shape of the position eigenfunction depends on the Hamiltonian.
    (2) The shape of the energy eigenfunction depends on the Hamiltonian.
    (3) No matter what kind of Hamiltonian the system has, the position eigenfunction is always a delta function in position space.

A. 1 only

B. 3 only

C. 1 and 2

D. 2 and 3

E. None of the above

24. Consider the following statement: If the initial state is $|\Psi\rangle$ for a particle in a 1-D infinite square well, $\left|\langle\psi_1|H|\Psi\rangle\right|^2$ is the probability of obtaining energy E_1 when measuring the energy of the particle. Do you agree with this statement? Explain. (Hint : Consider the unit of $\langle\psi_1|H|\Psi\rangle$.)

25. For a particle in a 1-D infinite square well, suppose its initial state is $\left|\Psi\right\rangle$. What are the physical meanings of $\left\langle\Psi\middle|H\middle|\Psi\right\rangle$ and $\left\langle\Psi\middle|x\middle|\Psi\right\rangle$?

Hint for question 9: In position representation, $\psi(x)=\sum_n A_n\psi_n(x)$***. Use Fourier trick. Multiple both sides by*** ψ_m^****, integrate over all space and use orthonormality of energy eigenstates. Note that*** $\int_{-\infty}^{+\infty}\psi_m^*(x)\Psi(x)dx=\sum_n A_n\int_{-\infty}^{+\infty}\psi_m^*\psi_n dx$ ***and*** $\sum_n A_n\delta_{mn}=A_m$ ***. Alternatively, in Dirac notation,*** $\left\langle\psi_m\middle|\psi\right\rangle=\sum_n A_n\left\langle\psi_m\middle|\psi_n\right\rangle=\sum_n A_n\delta_{mn}=A_m$***. We can use*** $\int_{all}\left|x\right\rangle\left\langle x\right|dx=1$ ***to write*** $\left\langle\psi_m\middle|\psi\right\rangle$ ***in position representation.***

Part 2 Quantum measurement and time evolution

For a particle in a 1-D infinite square well ($V(x)=0$ when $0<x<a$ and $V=+\infty$ elsewhere), the energies are $E_n=\dfrac{n^2\pi^2\hbar^2}{2ma^2}$ (n=1,2,3,…) and the energy eigenstate corresponding to each E_n is $\psi_n(x)=\sqrt{\dfrac{2}{a}}\sin\left(\dfrac{n\pi x}{a}\right)$ when $0<x<a$ and $\psi_n(x)=0$ otherwise. Answer the following questions. (Question 26~34)

26. At time $t=0$, the initial state of the particle is the ground state ψ_1. If we measure the energy of the particle at time t, what result(s) can we obtain?

A. Only E_1

B. Only $E_1 e^{-iE_1 t/\hbar}$

C. Any of the energies E_n, n=1,2,3,…

D. Any of $E_n e^{-iE_n t/\hbar}$, *n*=1,2,3,…

E. $\sum c_n E_n$, c_n are constants and at least two of c_n are non-zero, *n*=1, 2, 3, …

27. In the previous problem (*question 26*), after the measurement of energy, what state will the particle be in?

A. The ground state $|\psi_1\rangle$

B. Any of the states $|\psi_n\rangle$, *n*=1,2,3,…

C. $\sum_n A_n|\psi_n\rangle$, A_n are constants and at least two of A_n are non-zero, *n*=1,2,3,…

D. None of the above

Simulation 6

- Choose the simulation “psi1”. The initial state in this simulation is ψ_1. Next, click the triangular button (to start and stop the time evolution) on the lower left corner of the window. You can see a clock at the lower right corner of the window showing the time. Does the shape of the absolute value of wavefunction change with time? Why is an energy eigenstate called a “stationary state”?

- Now measure the energy around t=2 units. What is the state of the system after the energy measurement? Set back the simulation to the initial state $|\psi_1\rangle$ and measure the energy again around t=3 units. Is the result the same as your first measurement (around t=2 units)? Is this result consistent with your answer to *question 26*?

28. Suppose the initial state of the particle is the first excited state $|\psi_2\rangle$. When you measure the energy of the particle, is it possible to obtain the ground state energy E_1? Explain.

29. At time $t = 0$, the initial state of the particle is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. If we measure the energy of the particle after time t, what result(s) can we obtain?

A. $(E_1 + E_2)/2$

B. E_1 or E_2

C. $(E_1 e^{-iE_1 t/\hbar} + E_2 e^{-iE_2 t/\hbar})/2$

D. $E_1 e^{-iE_1 t/\hbar}$ or $E_2 e^{-iE_2 t/\hbar}$

E. Any of E_n, n=1,2,3,…

30. In the previous problem (*question 29*), right BEFORE the measurement of energy, what state will the particle be in?

A. $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$

B. $|\psi_1\rangle$ or $|\psi_2\rangle$

C. $\frac{1}{\sqrt{2}}(|\psi_1\rangle e^{-iE_1 t/\hbar} + |\psi_2\rangle e^{-iE_2 t/\hbar})$

D. $|\psi_1\rangle e^{-iE_1 t/\hbar}$ or $|\psi_2\rangle e^{-iE_2 t/\hbar}$

31. In the previous problem (*question 29*), after the measurement of energy, what state will the particle be in?

A. $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$

B. Either $|\psi_1\rangle$ or $|\psi_2\rangle$

C. Any of $|\psi_n\rangle$, n=1,2,3,…

D. $\sum_n A_n|\psi_n\rangle$, A_n is constant and at least two of A_n are non-zero, n=1,2,3,…

E. None of the above

Simulation 7

- Open the simulation "psi1+psi2". The initial state in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Start the time evolution. Does the shape of the absolute value of the wavefunction change with time? Is the state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ a stationary state?

- Now measure the energy around t=2 units. What is the state of the particle after the energy measurement? Suppose you obtain state $|\psi_i\rangle$ (i=1 or 2) in the first measurement of energy. If you set back the simulation to the initial state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ and measure the energy again around the same time t=2 units, do you think you have the same probability of obtaining $|\psi_i\rangle$ as in your first measurement? Does the probability of obtaining $|\psi_i\rangle$ change if you re-initialize the state and measure the energy around the time t=3 units? (Note that you only need to write down your conclusion and explanation without measuring the energy repeatedly to estimate the probability.)

32. At time $t=0$, the initial normalized state of the particle is $\sum A_n \psi_n$, where A_n are normalized non-zero constants. If we measure the energy of the particle at time t, what result can we obtain?

A. Any of E_n, n=1,2,3,…

B. Any of $A_n E_n$

C. Any of $A_n E_n e^{-iE_n t/\hbar}$

D. $\sum_n A_n E_n e^{-iE_n t/\hbar}$

E. $\sum_n |A_n|^2 E_n e^{-iE_n t/\hbar}$

33. In *problem 32*, right BEFORE the measurement of energy, what state will the particle be in?

A. $\sum_n A_n |\psi_n\rangle$

B. $|\psi_n\rangle$

C. $\sum_n A_n |\psi_n\rangle e^{-iE_n t/\hbar}$

D. $|\psi_n\rangle e^{-iE_n t/\hbar}$

34. In *problem 33*, what is the probability of measuring energy E_n?

(1) $A_n e^{-iE_n t/\hbar}$

(2) $|A_n|^2$

(3) $\left|A_n e^{-iE_n t/\hbar}\right|^2$

A. 1 only B. 2 only C. 3 only D. 2 and 3 only E. all of the above

Simulation 8

- Choose the simulation "psi1+psi_n". The initial state in this simulation is $\sum_n A_n|\psi_n\rangle$. Start the time evolution. Does the shape of the absolute value of the wavefunction change with time?

- Reset the simulation to the initial state $\sum_n A_n|\psi_n\rangle$ and make an energy measurement at time *t*=0. Sketch the wave function you observed in the simulation. Which energy eigenstate $|\psi_i\rangle$ do you obtain? Which energy have you measured?

- Now reset the simulation to the initial state $\sum_n A_n|\psi_n\rangle$ and start the time evolution. Measure the energy around *t*=2 units. What is the state of the particle after the energy measurement? What is the energy that you measured? Write down how the state $\sum_n A_n|\psi_n\rangle$ evolves with time and calculate the probability of measuring energy E_n. Does the probability of measuring a particular energy E_n and collapsing into an energy eigenstate $|\psi_n\rangle$ change with time? Explain. (You only need to write down your conclusion and explanation without measuring the energy repeatedly to estimate the probability.)

For a particle interacting with a simple harmonic oscillator (SHO) potential, the allowed energies are $E_n = (n+\frac{1}{2})\hbar\omega$ *(n=0,1,2,…), and the energy eigenstate corresponding to each* E_n *is* $\psi_n(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$, *where* $H_n(\xi)$ *is the* n^{th} *Hermite polynomial and* $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$ *is a dimensionless variable. The first three Hermite polynomials are* $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$. *Answer the following questions (questions 35 & 36).*

Measurement of the energy of SHO at time t>0.

35. At time $t=0$, suppose the initial state of a simple harmonic oscillator is a Gaussian function $\psi(x)=Ae^{-\xi^2/2}$, where A is a positive constant. If we measure the energy of the simple harmonic oscillator at time t, what result can we obtain?

A. $E=\hbar\omega$ only

B. $E=\sum_{n=0}^{\infty}(n+\frac{1}{2})\hbar\omega$, n=0, 1, 2, …

C. $E=\frac{1}{2}\hbar\omega$ only

D. Any of the energies $E_n=(n+\frac{1}{2})\hbar\omega$, n=0,1,2, …

36. In the previous problem (*question 35*), after the measurement of energy, what state will the particle be in?

A. $\psi(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$

B. $\psi(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$, n=1,2,3,…

C. $\psi(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(x)e^{-x^2/2}$, n=0 only

D. Any of $\psi(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(x)e^{-x^2/2}$, n=1,2,3,…

E. $\psi(x)=\sum c_n\psi_n$, where $\psi_n(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(x)e^{-x^2/2}$, n=0,1,2,…

Now consider a particle in a 1-D infinite square well ($V(x)=0$ when $0<x<a$ and $V(x)=+\infty$ otherwise). Answer the following questions (*question 37-40*)**.**

37. Suppose at time $t=0$ the initial state wavefunction of the particle is $\Psi(x)=Ax(a-x)$ for $0<x<a$ and $\Psi(x)=0$ otherwise. If you measure the energy of the particle at time t, what is the probability of obtaining E_n? You can leave the probability as an integral. (Hint: Recall *question 9* in the first part of this tutorial. You can write $\Psi(x)$ in the basis of energy eigenfunctions as $\Psi(x)=\sum_n A_n\psi_n$ and find the coefficients A_n by projecting the state $|\Psi\rangle$ along the energy eigenstate $|\psi_n\rangle$ or by using the Fourier trick.)

38. Given the wavefunction at time $t=0$, why is it useful to write the state of a quantum system as a superposition of energy eigenstates to find the wavefunction after time *t*? (The answer is on the last page of the tutorial.)

<u>*Measurement of postion*</u>

39. Harry and Sally prepare the same initial wavefunction $\frac{\psi_1(x)+\psi_2(x)}{\sqrt{2}}$ which is a linear superposition of the energy eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ in their labs at time *t*=0. They each make a measurement of the position of the electron after different time *t*. The wave function at time *t* is $\Psi(x,t)=\frac{\psi_1(x)e^{-iE_1t/\hbar}+\psi_2(x)e^{-iE_2t/\hbar}}{\sqrt{2}}$. Harry measures the position of his electron at time *t*=1 unit and Sally measures the position of her electron at time *t*=3 units. Consider the following conversation between Harry and Sally.

Harry: The probability that I will find my electron between x_0 and x_0+dx is not the same as the probability that you will find your electron between x_0 and x_0+dx. The probability is determined by the absolute square of the wave function, $\left|\Psi(x_0,t)\right|^2 dx$, which depends on time.

Sally: I agree that the probability density for measuring position depends on $\left|\Psi(x_0,t)\right|^2$. But when you calculate $\left|\Psi(x_0,t)\right|^2$, the time dependent phase factors will cancel out and the probability density will be time independent. You and I have the same probability of measuring the position between x_0 and x_0+dx.

Harry: The time-dependent phase factors do not drop out of the cross terms. We need to square the whole wave function, not only the coefficients of $\psi_1(x)$ and $\psi_2(x)$ separately. That is why we get time dependent cross terms.

With whom do you agree? Explain. Use the simulation "psi1+psi2" to justify your answer. *(In this simulation, the position eigenfunction is drawn as a narrow function (but not a delta function) due to constraints in the simulation. It is an approximation for a delta function obtained in an ideal position measurement which has an infinitely high peak and infinitesimal width.)*

Measurement of energy

40. Harry and Sally prepare the same initial state wavefunctions $\frac{\psi_1(x)+\psi_2(x)}{\sqrt{2}}$ from energy eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ in their labs at time *t*=0. Harry measures the energy of his electron in a 1D infinite square well at t=1 unit and Sally measures the energy of her electron in an identical 1D infinite square well at time t=3 units. Consider the following conversation between Harry and Sally.

Harry: The probability that I will measure energy E_n is not the same as the probability that you will measure energy E_n. The probability is determined by the absolute square of the wavefunction, $|\Psi(x,t)|^2$, which depends on time.

Sally: No. The probability of measuring **position** depends on the absolute square of the wave function. This time we are measuring **energy**. The time-dependent phase factors will cancel out because only one factor $e^{-iE_nt/\hbar}$ can contribute in calculating the probability of measuring a particular energy E_n. Thus, the probability of obtaining E_n will be time independent. You and I have the same probability of measuring energy E_n.

Harry: But there will be cross terms in the square of the wave function. The phase factors do not drop out for the cross terms.

Sally: I disagree. The probability of measuring energy is determined by the square of the coefficients of each of the energy eigenfunctions $\psi_1(x)$ and $\psi_2(x)$. We do not square the entire wavefunction, we only square the coefficients of each energy eigenfunction and the time dependence drops out. For example, the probability of measuring energy E_1 is given by: $p(E_1)=\left|\frac{e^{-iE_1t/\hbar}}{\sqrt{2}}\right|^2=\frac{1}{2}$, which is time independent.

With whom do you agree? Explain.

Simulation 9* (Complete if time is available)
If you are not sure about the answer to question 40, you may check it with the simulation. Measure the energy at t=1 unit for 20 trials, and estimate the probability of obtaining E_1. Then measure the energy at t=3 units for 20 trials and estimate the probability of obtaining E_1. Combine your data with other groups' to make the result statistically reliable.

Consecutive measurements
Measure the energy of the system first and then measure the energy again.

41. At time $t = 0$, the initial state of the particle is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. We first measure the energy of the particle at time t and obtain the energy E_1. Then we immediately measure the energy again. What result can we obtain in the second measurement? Explain your choice.

A. Only E_1

B. Either E_1 or E_2

C. Only $E_1 e^{-iE_1 t/\hbar}$

D. Either $E_1 e^{-iE_1 t/\hbar}$ or $E_2 e^{-iE_2 t/\hbar}$

E. Any of E_n, n=1,2,3, …

42. In the previous problem (*question 41*), after the second measurement of energy, what state will the particle be in?

A. $|\psi_1\rangle$ or $|\psi_2\rangle$

B. Any of $|\psi_n\rangle$, n=1,2,3,…

C. $|\psi_1\rangle$

D. $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$

E. None of the above

Simulation 10

Choose the simulation “psi1+psi2”. The initial state in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Start the time evolution. Around *t*=2 units, first click the start/stop button to pause the time evolution and then measure the energy. What state do you obtain? Then measure the energy again **without** re-initializing the wavefunction. Is the state the same as the state you observed after your first measurement?

43. In *question 41*, if the time interval between the first and second energy measurement is $\Delta t > 0$, what is the measured energy and state of the particle after the second measurement?

Simulation 11

Choose the simulation “psi1+psi2”. The initial state in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Start the time evolution. Around *t*=2 units, first click the start/stop button to pause the time evolution and then measure the energy. What state do you obtain? Then start the time evolution and measure the energy again at *t*=3 units without re-initializing the wavefunction. Is the state the same as your first measurement? (*Note that the clock would return to zero when you restart the time evolution.*)

First measure the energy of the system and then measure the position after the energy measurement.

44. At time $t = 0$, the initial state of the particle is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. We first measure the energy of the particle at time $t = t_0$ and obtain the result E_1. Then we immediately measure the position of the particle (also at time $t = t_0$). What is the probability of finding the particle in the region between x_0 and $x_0 + dx$?

45. In the previous problem (*question 44*), if the measurement of position is made at $t = t_1$ instead of t_0 (not immediately after the energy measurement), what is the probability of finding the particle in the region between x_0 and $x_0 + dx$? If the particle is found at $x = x_0$, what is the state of the particle after the position measurement in Dirac notation and in the position representation?

Simulation 12

- Choose the simulation "psi1+psi2". The initial state in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$.

  Start the time evolution. Around *t*=2 units, first click the start/stop button to pause the time evolution and then measure the energy. What state do you obtain? Explain

- Start the time evolution. Does the shape of the absolute value of wave function change with time? According to your wave function, what is the most probable position for finding the particle? Does this most probable position change with time? Explain

First measure the position of the system and then measure the position again.

46. We first measure the position of a particle in a 1-D infinite square well at time $t = 0$ and find the particle at $x = x_0$. At time *t*(>0) after the position measurement, what state will the particle be in? Write your answer in terms of an expansion in a complete set of energy eigenstates. Use ψ_n and E_n to denote the energy eigenstates and energy eigenvalues. (Hint: refer to *question 37*)

47. In *question 46*, when we make a second measurement of position at time t(>0), what is the probability density of finding the particle at $x = x_0$? Does the probability density depend on time t when the measurement was performed?

48. In *question 46*, if the second measurement of position is made immediately after the first position measurement at time $t = 0$, what is the probability density of finding the particle at $x = x_0$?

Simulation 13

- Choose the simulation "psi1+psi2". The initial state in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$.

  Start the time evolution. Around t=2 units, first click the start/stop button to pause the time evolution and then measure the position. What state do you obtain? Then measure the position again after the first position measurement without starting the time evolution, will you obtain the same state as the first position measurement? Explain.

- Then start the time evolution. Does the shape of wave function change with time? Will the wave function go back to the state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$? According to your wave function at t=10 units, what is the most probable position for finding the particle? Does this most probable position change with time? Explain.

(*Note that if you make the second position measurement immediately after the first position measurement, you may find that the wavefunction after the second measurement shifts its position somewhat. This is because the wavefunction in which the system collapses after the position measurement in our simulation is not an ideal position eigenfunction (it is not a delta function). If we had a delta function, the position eigenfunction would be highly localized and the second measurement of position in immediate succession would give us the same result as the first position measurement.*)

First measure the position of the system and then measure the energy after the position measurement.

49. Suppose we measure the position of a particle for the initial state $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ in a 1-D infinite square well at time $t = 0$ and find the particle at $x = x_0$. Then we measure the energy of the particle immediately after the position measurement. What is the probability of obtaining the ground state energy? (Hint: In order to find the probability of measuring energy, the wavefunction must be expanded in term of a complete set of energy eigenstates.)

50. In *question 49*, if we perform the measurement of energy after time $t = t_0$, what is the probability of measuring the ground state energy? Is the result the same as the result for the immediate energy measurement? (Hint: Find the wavefunction after time *t* and then calculate the probability of measuring ground state energy.)

Simulation 14

- Choose the simulation "psi1+psi2". The initial state in this simulation is $\frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$. Start the time evolution. Around *t*=2 units, first click the start/stop button to pause the time evolution and then measure the position. Draw the shape of the wavefunction you obtain after the position measurement. Is this what you expected?

- Then measure the energy without restarting the time evolution. Can you predict what energy you will obtain (which energy eigenstate your system will collapse to) after the energy measurement? Will you obtain any energy eigenstates other than $|\psi_1\rangle$ and $|\psi_2\rangle$? Explain.

Answer to *question 38*

The Hamiltonian governs the time evolution of the system according to the time dependent Schrödinger equation (TDSE). Since energy eigenstates $|\psi_n\rangle$ *are eigenstates of the* $\hat{H}$ *operator, the energy eigenstates have a simple time evolution of the form* $|\psi_n\rangle e^{-iE_n t/\hbar}$. *When we write a general state as a superposition of the energy eigenstates (or stationary states), each term in the superposition evolves according to a different phase of the type* $e^{-iE_n t/\hbar}$ *(assuming no degeneracy) so that the state at time t is* $\sum_n A_n |\psi_n\rangle e^{-iE_n t/\hbar}$ *where* A_n *can be calculated by using the Fourier trick in position representation or by projecting the initial state along the energy eigenstate* $|\psi_n\rangle$.