Solution to the Reflective Homework

(1) The generic form for a wave packet for a free particle at time $t = 0$ is given by

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \tag{1}$$

At time $t = 0$, we construct two different Gaussian wave packets for the free particle: One is $\Psi_1(x, t = 0) = A_1 e^{-x^2/(2\sigma_1^2)}$ which can be written as $\Psi_1(x, t = 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_1(k) e^{ikx} dk$ and another is $\Psi_2(x, t = 0) = A_2 e^{-x^2/(2\sigma_2^2)}$ which can be written as $\Psi_2(x, t = 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_2(k) e^{ikx} dk$. Here A_1 and A_2 are the normalization constants and σ_1 and σ_2 are the standard deviations of the two wave packets respectively. Note that $\phi_1(k)$ is the Fourier transform of $\Psi_1(x, t = 0)$ and $\phi_2(k)$ is the Fourier transform of $\Psi_2(x, t = 0)$. Thus, $\Psi_1(x, t = 0)$ and $\Psi_2(x, t = 0)$ are the position space wave functions and $\phi_1(k)$ and $\phi_2(k)$ are the corresponding momentum space wave functions. From the uncertainty principle for position and momentum, $\sigma_x \sigma_p \geq \hbar/2$ and for a Gaussian wave packet $\sigma_x \sigma_p = \hbar/2$ (limiting case). Here, σ_x is the standard deviation for the position measurement and can be obtained from the position space wave function and σ_p is the standard deviation for the momentum measurement and can be obtained from the momentum space wave function. Since $\sigma_1 > \sigma_2$ for the position space wave functions given, uncertainty principle implies that $\sigma_{1p} < \sigma_{2p}$ (relation between the standard deviation of the two momentum space wave functions). Since the Fourier transform of a Gaussian function is also a Gaussian function, each of $\phi_1(k)$ and $\phi_2(k)$ is also Gaussian. However, the width or the standard deviation of $\phi_1(k)$ will be less than that for $\phi_2(k)$ from the uncertainty principle ($\sigma_{1p} < \sigma_{2p}$ and $p = \hbar k$).

Note: The above results can be obtained simply by noting that if the standard deviation for Gaussian function 1 in position space is larger than Gaussian function 2, then the Fourier transform of function 1 will have a smaller standard deviation than the Fourier transform of function 2. This reflective problem helps you realize that it is a manifestation of the uncertainty principle.

(2) The generic form for a wave packet for a free particle at time t is given by $\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega(k)t)} dk$. where $\omega(k)$ is the angular frequency which is a function of the magnitude of the wave vector k. Suppose $\phi(k)$ in the above equation is <u>not</u> highly localized about any particular k value. In that case, we cannot linearize the dispersion relation which is the relation between the angular frequency ω and the magnitude of the wave vector k (or the relation between energy and momentum). If ω and k are not linearly related, the group velocity of the wave packet will not be well defined. The shape of the wave function will also change with time.

(3) The particle in a classical bound state will have two classical turning points on the two sides (where the total energy of the particle equals the potential energy so that the particle's kinetic energy is zero). Quantum mechanically, the particle can tunnel through a potential barrier and has a non-zero probability of being found in a classically forbidden region. Therefore, the criterion for the particle being in a bound state does not involve classical turning points but the particle's energy should be less than the potential energy at both $\pm\infty$.

(4) See Griffiths for a diagram of a particle interacting with a potential energy which is classically in a bound state but quantum mechanically in a scattering state. No, it is not possible to have a particle interacting with a potential energy which is quantum mechanically in a bound state but classically in a scattering state. If the particle's energy is less than the potential energy at both $\pm\infty$ then there must be two classical turning points on the two sides and the particle is also going to be classically bound.